Lagrangian formalism in the cosmological perturbation theory

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Abstract

There is given a short overview of gauge-invariant formalism which was developed in [1][2][4]. Equivalency of hydrodynamical and field approach is shown.

1 Introduction

The problem of considering cosmological perturbations within the scope of General Relativity has been treated since the classical work [10]. While considering perturbations one uses a certain gauge, e.g. synchronous ($g_{00} = 1,\ g_{oi} = 0$) or comoving (where the energy flux vanishes), but the spirit of GR where all reference frames are equivalent enforces us to show the transformation from one gauge to another. And these transformation formulae may be ambiguous. For example, transformation to synchronous gauge contains two integration constants. This ambiguity reflects the general fact that splitting of cosmological quantities (the metric tensor, scalar field, energy density etc.) into background and perturbation part in an arbitrary gauge is not unique. It can cause appearing of unphysical perturbation modes which does not affect physical quantities at all. One of the ways to solve it is constructing and working only with gauge-invariant variables [1][2][4].

Equation which describes the evolution of a gauge-invariant variable comes from the perturbed part of Einstein equations. Clearly, Einstein equations solely are insufficient, because perturbations of both the metric tensor and the energy-momentum tensor can be written in a general form without specifying any physics. In order to obtain a dynamical equation we need a physical relation. For example, it can be a relation between energy density and pressure perturbations which corresponds to the *hydrodynamical* approach. Also we can start from a general Lagrangian of φ-field (which serves as a 3-velocity potential), and this corresponds to the *field* approach. The former is usually associated with equation of state, while the latter is with models of inflation. We show that both approaches are equivalent.

The plan of the paper is as follows. In Sec. 2 there is given a short review of gauge-invariant formalism, in Sec. 3 it is shown equivalency of the hydrodynamical and field approaches. In Sec. 4 we relate different gauge-invariant variables which are used most often.

2 An overview of gauge-invariant language

Below we work with the background Fridmann-Robertson-Walker metric:

$$ds^2 = dt^2 - a^2 dx_i dx^i = a^2(d\eta^2 - dx_i dx^i). \tag{1}$$

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The background 4-velocity is $u^\mu = (1,0,0,0)$, speed of light $c = 1$, and η is conformal time, $d\eta = dt/a$.

The basic equations we need below are the Fridmann equations:

$$H^2 = \frac{8\pi G}{3}\varepsilon, \tag{2}$$

$$\gamma \equiv -\frac{\dot H}{H^2} = \frac{3}{2}\left(1+\frac{p}{\varepsilon}\right), \tag{3}$$

where $H = \dot a/a$, a is the scale factor, and G is the gravitational constant. The dot and the prime stand for the derivative with respect to physical time t and conformal time η, respectively.

Generally, scalar type metric perturbations are constructed using four "potentials" [5]: A, B, C and D:

$$h_{\mu\nu} = \left(\begin{array}{cc} 2D & aC_{,i} \\ aC_{,i} & 2a^2(A\delta_{ij} + B_{,ij}) \end{array}\right). \tag{4}$$

The potential A is actually a perturbation of the scale factor: $A = -\delta a/a$.

The perturbation of the energy-momentum tensor looks as follows:

$$\delta T^0_0 = \delta\varepsilon,$$

$$\delta T^0_i = (\varepsilon + p)v_{,i}, \tag{5}$$

$$-\delta T^i_j = \delta p \delta_{ij} + (\varepsilon + p)\sigma_{ij},$$

$$\sigma_{ij} = (aH)^{-2}(E_{,ij} - \triangle E\delta_{ij}), \quad \sigma^j_{i,j} = 0,$$

where v is the 3-velocity potential ($u_i = v_{,i}$), $u_\mu = (1, u_i)$, and E presents anisotropic stresses.

Thus, we have four gravitational potentials A, B, C, D and four matter potentials v, $\delta\varepsilon$, δp, E. All of them but E are not gauge-invariant. By small coordinate transformations $x^\mu \rightarrow x^\mu + \xi^\mu$ the potentials get changed. Two of these eight potentials are arbitrary. It corresponds to a gauge choice (an arbitrary vector in scalar representation $\xi_\mu = F u_\mu + H_{,\mu}$). It is possible to construct some gauge-invariant combinations of the potentials. All such combinations constitute an infinite set.

The potentials A, B, C, D, v, $\delta\varepsilon$, δp, E are not independent. They are linked through the first-order expansion of the Einstein equations

$$\delta G^\mu_\nu = 8\pi G \delta T^\mu_\nu. \tag{6}$$

The natural gauge-invariant combination is that of the gravitational potential A and the velocity potential v which is called the q-scalar [1]:

$$q = A + Hv. \tag{7}$$

The inverse transformations of the q-field to the original potentials are as follows:

$$v = \frac{q - A}{H}, \quad \delta p_c \equiv \delta p - \dot p v = \frac{\varepsilon + p}{H}\dot q, \tag{8}$$

$$a\dot B - C = \frac{A - \Phi_H}{aH}, \quad D = \gamma q - \frac{d}{dt}\left(\frac{A}{H}\right), \tag{9}$$

$$\delta\varepsilon_c \equiv \delta\varepsilon - \dot\varepsilon v = \frac{\triangle\Phi_H}{4\pi G a^2}, \quad \Phi_H = \frac{H}{a}\int a\gamma(q + 2E)dt, \tag{10}$$

where δp_c and $\delta\varepsilon_c$ are gauge-invariant variables which coincide with pressure and energy density perturbation in the comoving reference frame, respectively. The first equation in (10) is, in fact, the relativistic Poisson equation. From the inverse transformations it can explicitly be seen that E is a gauge-invariant potential. The potential Φ_H was firstly introduced by Bardeen [2].

3 Introducing dynamics

The previous analysis is common and does not depend on any concrete physics. However, in order to introduce dynamics we need some additional relation, e.g. between δp_c and $\delta\varepsilon_c$, that is, we need to specify some physics. *We have two possibilities: either we can use the hydrodynamical approach to relate δp_c and $\delta\varepsilon_c$ or the field approach, i.e. to admit some form of the Lagrangian. Further it is shown that the both approaches are equivalent in this problem.* Further on we suppose absence of anisotropic stresses, therefore,

$$\begin{gathered} E=0, \\ \Phi_H=\frac{H}{a}\int a\gamma q dt. \end{gathered} \tag{11}$$

Hydrodynamical approach. In the hydrodynamical approach we assume

$$\delta p_c=\beta^2(t)\delta\varepsilon_c, \tag{12}$$

where $\beta(t)$ is a function of time. Hence from (8) and (3),

$$\delta\varepsilon_c=\alpha^2 H\dot{q}, \quad \alpha^2=\frac{\gamma}{4\pi G\beta^2}. \tag{13}$$

Relation (12) means that there is only one medium and we describe its perturbations. As soon as (12) is valid equations (10) and (13) immediately give:

$$\gamma\beta^{-2}a^3\dot{q}=\int a\gamma\triangle q dt. \tag{14}$$

After differentiation the last equation gives equation describing the evolution of q-scalar:

$$\ddot{q}+\left(3H+2\frac{\dot{\alpha}}{\alpha}\right)\dot{q}-\left(\frac{\beta}{a}\right)^2\triangle q=0. \tag{15}$$

The equation (15) corresponds to the action

$$\begin{aligned} S[q] &= \frac{1}{2}\int \alpha^2\left(\dot{q}^2-\left(\frac{\beta}{a}\right)^2 q_{,i}q^{,i}\right)a^3 dt d^3x = \\ &= \frac{1}{2}\int (\alpha a)^2\left(q'^2-\beta^2 q_{,i}q^{,i}\right)d\eta d^3x. \end{aligned} \tag{16}$$

Since the backward path from equation to a Lagrangian defines the Lagrangian to a factor before it, we can see that (16) has the right coefficient if we look at it in some asymptotic limit, e.g. in the limit of small scales (the sound wave frequency $\omega \gg H$ and sound velocity $c_s\simeq\beta\simeq 1$). In this approximation $q\simeq H\upsilon$, $\dot{q}\simeq H\dot{\upsilon}$ and $\delta\varepsilon_c\simeq\delta\varepsilon$. Using the relations:

$$\frac{\delta\varepsilon}{\varepsilon+p}=\frac{\dot{\upsilon}}{c_s^2}, \quad \frac{\nabla\upsilon}{a}=-\mathbf{v}, \tag{17}$$

where $\mathbf{v}$ is hydrodynamical velocity in a sound wave, we have the following chain of equalities:

$$\begin{aligned} L[q] &= \frac{a^3\alpha^2}{2}\left(\dot{q}^2-\left(\frac{\beta}{a}\right)^2 q_{,i}q^{,i}\right)=\frac{\gamma a^3 H^2}{8\pi G c_s^2}\left(\dot{\upsilon}^2-\frac{c_s^2}{a^2}(\nabla\upsilon)^2\right)= \\ &= \frac{a^3}{2}\left((\varepsilon+p)\frac{\dot{\upsilon}^2}{c_s^2}-(\varepsilon+p)\mathbf{v}^2\right)=\frac{a^3}{2}\left(c_s^2\frac{\delta\varepsilon^2}{\varepsilon+p}-(\varepsilon+p)\mathbf{v}^2\right). \end{aligned} \tag{18}$$

The corresponding comoving volume energy density is

$$\mathcal{E} = \frac{1}{2}\left(c_s^2 \frac{\delta\varepsilon^2}{\varepsilon + p} + (\varepsilon + p)\mathbf{v}^2\right). \tag{19}$$

The last expression is exactly the energy density in a sound wave [6], [1].

Field approach. The Universe filled with a scalar field φ. The relation to the 4-velocity is $u_\mu = \varphi_{,\mu}/w$, where $w^2 = \varphi_{,\mu}\varphi_{,\nu}g^{\mu\nu}$. Below we follow the approach of [1], [9].

The Lagrangian of the scalar field can be taken in a quite arbitrary form:

$$\mathcal{L} = \mathcal{L}(\varphi, w). \tag{20}$$

After that we obtain the energy-momentum tensor:

$$\varepsilon = nw - \mathcal{L}, \qquad p = \mathcal{L}, \qquad n = \frac{\partial \mathcal{L}}{\partial w}, \tag{21}$$

and

$$\upsilon = \frac{\delta\varphi}{\dot{\varphi}}, \qquad \frac{\delta\varepsilon_c}{\varepsilon + p} = \frac{\dot{q}}{c_s^2 H}, \qquad c_s^{-2} = \frac{w}{n}\frac{\partial^2 \mathcal{L}}{\partial w^2}. \tag{22}$$

This proves that in this case $c_s \equiv \beta$, and, thus, both ways, (12) and (20), of deriving equation (15) are identical.

In order to obtain Lagrangian describing perturbations we need to expand the action for gravitating scalar field to the second order in perturbation. The action is standard:

$$S[\varphi, g_{\mu\nu}] = \int (\mathcal{L} - \frac{1}{16\pi G} R)(-g)^{1/2} d^4x, \tag{23}$$

where R is scalar curvature.

The first step is to perturb the metric tensor: $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ and the scalar field: $\varphi \to \varphi + w\upsilon$. First order terms turn to zero since the background Einstein equations are satisfied. The result of the expanding the action (23) to the second order in perturbation is as follows (total divergency terms are omitted):

$$\begin{aligned}
\delta^{(2)} S = &-\frac{1}{64\pi G}\int (\overline{h}_{\sigma\beta;\alpha}\overline{h}^{\sigma\beta;\alpha} - 2\overline{h}^{\alpha\beta}{}_{;\sigma}\overline{h}^{\sigma}{}_{\alpha;\beta} - \frac{1}{2}\Box\overline{h})(-g)^{1/2} d^4x - \\
&- \frac{1}{4}\int (\frac{1}{16\pi G} R - \mathcal{L})(\overline{h}^{\mu}_{\nu}\overline{h}^{\nu}_{\mu} - \frac{1}{2}\overline{h}^2)(-g)^{1/2} d^4x + \\
&+ \frac{1}{2}\int nw\left[\nu\upsilon\overline{h} - 2u_\mu\upsilon_\nu\overline{h}^{\mu\nu} + \chi^2(c_s^{-2} - 1) + m^2\upsilon^2 + 2\Gamma\upsilon\chi\right](-g)^{1/2} d^4x,
\end{aligned} \tag{24}$$

$$n\nu = -\frac{\partial\mathcal{L}}{\partial\varphi}, \quad \Gamma = \frac{w}{n}\frac{\partial^2\mathcal{L}}{\partial w \partial\varphi}, \quad m^2 = -\frac{w}{n}\frac{\partial^2\mathcal{L}}{\partial\varphi^2}, \quad \chi = \frac{\delta w}{w}, \quad \upsilon_\mu = \frac{(w\upsilon)_{,\mu}}{w}.$$

Here $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$, is the so-called tensor with inverse trace: $\overline{h} = \overline{h}^{\sigma}{}_{\sigma} = -h = -h^{\sigma}{}_{\sigma}$. All operations of raising and lowering indices are performed using the background metric $g_{\mu\nu}$.

The variable q is so remarkable, because after linking all the potentials through equations (8) (9) (10) and substituting the q-scalar (7) to the expansion (24) we get a very simple perturbation action (16) (totally divergent terms are excluded), where q enters as a test massless-like field.

4 Relation between q, ζ and Ψ

In notations of [7] the variable q [1] looks as follows:

$$q = \Psi + Hv^{(gi)}, \tag{25}$$

where

$$\Psi = A - \frac{(B' - C)a'}{a} \tag{26}$$

and

$$v^{(gi)} = v + a(B' - C), \quad v^{(gi)}_{,i} = \delta u^{(gi)}_i. \tag{27}$$

The identity of (25) and (7) is obvious if we substitute (26) and (27) to (25).

Among others a variable ζ was introduced by Bardeen [3]. It is expressed through Φ_H as follows:

$$\zeta = \frac{2}{3}\frac{H^{-1}\dot{\Phi}_H + \Phi_H}{1 + w_B} + \Phi_H, \; w_B = \frac{p}{\varepsilon}. \tag{28}$$

Since (3)

$$\frac{1}{\gamma} = \frac{2}{3}\frac{1}{1 + w_B},$$

hence,

$$\zeta = \frac{H^{-1}\dot{\Phi}_H + \Phi_H}{\gamma} + \Phi_H = \frac{1}{a\gamma}\frac{d}{dt}\left(\frac{a\Phi_H}{H}\right) = q. \tag{29}$$

Obviously ζ coincides with q introduced in a general form by equation (7): $\zeta = q$.

5 Conclusion

The theory of primordial cosmological perturbations can be constructed in a quite general form. We needed the physical suggestions only at the last stage to link δp_c and $\delta\varepsilon_c$, and, thus, to enclose the set of equations. The hydrodynamical suggestion (12) and the field one (20) are usually understood as two separate things, the former being referred to an existing equation of state, and the latter associated with an inflaton field and, hence, models of inflation. However, it appears that the suggestion (12) is equivalent to the considering of a field φ with a quite common Lagrangian $\mathcal{L}(\varphi, w)$. In linear perturbation theory of a single gravitating medium the two approaches coincide.

As a side result of a short overview accomplished in Sections 1 and 4, it was shown that the variable ζ introduced in [3] is equal to the variable q introduced by Lukash [1], and equation (5.22) in [7] is equivalent to (15).

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