Entropy Growth and the Dark Energy Equation of State*

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Abstract

We revisit the conjecture of a generalized second law of thermodynamics which states that the combined entropy of matter and horizons must grow. In an expanding universe a generalized second law restricts the equation of state. In particular, it conflicts with long phases of a phantom, $w < -1$, equation of state.

1 Introduction

Observations indicate that our universe is in a phase of accelerated expansion [1–3]. Some mysterious dark energy appears to drive this acceleration. There exist various attempts to explain this phenomenon, most notably the (in)famous cosmological constant [4; 5] and a scalar field dubbed "quintessence" [6–8]. Both contribute a "dark energy" component to the cosmic energy density which can be described by a perfect fluid with an equation of state $w = \frac{p}{\rho} = \frac{T-V}{T+V}$.

Accelerated expansion requires $w < -\frac{1}{3}$. It is readily seen that for a positive potential V and positive kinetic energy T, one has $-1 \leq w \leq 1$. The cosmological constant, having no kinetic energy, has $w = -1$. Recently, however, also the case $w < -1$ has attracted a lot of attention, because it appears slightly favored by supernovae data [9]. A typical realization of $w < -1$ is provided by a scalar field with an inverted sign for the kinetic term [10][1].

In this note we consider restrictions on the dark energy equation of state suggested by the generalized second law of thermodynamics (GSL) [12–18],

$$dS = dS_{\mathrm{mat}} + 2\pi dA_{\mathrm{H}} \geq 0 \ , \tag{1}$$

where S_{mat} is the entropy of matter and A_{H} is the horizon area (we use units with $\hbar = c = 8\pi G = 1$). For vanishing dA_{H}, Eq. (1) is the well known second law of ordinary thermodynamics.

How can one understand the need for the additional contribution from horizons? This has first been discussed by Bekenstein for the case of a black hole [12]. If matter carrying entropy falls into a black hole it is hidden from the outside observer by its horizon, the surface that separates the regions from which light rays can/cannot reach an infinitely distant observer. Classically, the black hole appears as a very well ordered object, fully characterized by its mass, charge and angular momentum. Naively, this should correspond to a state of very low entropy.

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[1] This leads to instabilities which may be ameliorated for low energy effective theories with a sufficiently low cutoff [11].

The outside observer would therefore conclude that the observable entropy decreases, $dS_{\rm mat} < 0$, seemingly in violation of the second law of thermodynamics. It is very intriguing, however, that the horizon of a black hole grows when matter is added. Hence, one may conjecture that a generalized second law, Eq. (1), holds. Indeed, the ignorance of the observer, and therefore the entropy he affiliates with the system, increases because he cannot see what is behind the horizon.

Horizons, similar to the one of a black hole, also appear in cosmology in case of accelerated expansion, most notably for de Sitter space [19]. Shortly after the work of Bekenstein, Gibbons and Hawking suggested that also the future horizon of de Sitter space contributes to the total entropy in the same way as the black hole event horizon [14], which naturally leads to a generalized second law, Eq. (1), also for de Sitter space [17; 18]. Now the observer is inside the horizon and the horizon entropy represents the lack of information about the outside region which he cannot see. This conjecture has been examined in a variety of cases [17; 20–22]. However, the cosmological situations are more delicate than the case of a black hole. Except for de Sitter space, the horizon is not stationary, which implies a departure from thermal equilibrium. This is related to an apparent ambiguity in the choice of the "horizon" for which the GSL may hold. Consider the following three possibilities, which all have identical area for de Sitter space (we always assume spherical symmetry):

- The future or event horizon, which separates the regions from where light rays can/cannot reach an observer located at the center.
- The Hubble or apparent horizon, which is the surface moving away from a centrally located observer with the speed of light.
- The boundary of the causal region, with which an observer can communicate by sending a light ray and receiving the returned signal[2].

All of these surfaces have appealing and less appealing features. The future horizon, for example, is a true horizon which separates regions from which a central observer can/cannot receive information. However, it is not local in time and requires knowledge of the complete future evolution of the universe. In particular, one needs to know the equation of state until the infinite future to calculate the future horizon. The Hubble horizon, on the other hand, depends only on the current state of the universe. Yet, it is not a true horizon; in many situations one can receive light rays from outside the current Hubble horizon.

A related ambiguity concerns the volume used to calculate the matter entropy appearing in Eq. (1). Possible choices include (cf. [19; 23])

- The space-like volume "inside" the horizon.
- The light-like hypersurface defined by light rays starting from the horizon going "in"[3].
- The light-like hypersurface given by light rays originating in the center.

Fortunately, the results for the different types of volumes do not differ, as long as we assume that no matter entropy is generated, as we will in this note.

¿From the above discussion it is clear that for general cosmological situations the status of the GSL is that of a conjecture backed up by some examples, where essential aspects remain to be clarified. Nevertheless, in the following we will go ahead, apply the GSL and see what it can tell us about the dark energy equation of state. To be explicit we will discuss two versions of the

[2] Here, we measure the area of the surface at which the last light rays that asymptotically return to the oberver are reflected. This surface is the future horizon at a later time. Since in de Sitter space the area of the future horizon is constant the areas of the different horizons are equal.

[3] If one wants to ensure that the light rays reach the centrally located observer one may have to start a tiny bit away from the horizon.

GSL, with the future horizon and the Hubble horizon, respectively (for more details see [22]). Furthermore, we will restrict the discussion to a flat universe as suggested by observations.

As anticipated in [18; 24; 25], and shown in some detail later on, superaccelerated expansion resulting from $w < -1$ typically is in conflict with Eq. (1). The basic reason is very simple. Consider the case where the entropy of the horizon is the dominating contribution to the total entropy[4]. During a phase of accelerated expansion the distance to the horizon is $\sim \frac{1}{H}$ (future horizon and Hubble horizon are roughly of the same size) and its area is $A_{\rm H} \sim \frac{1}{H^2}$. Since the superaccelerated expansion is characterized by $\dot{H} > 0$, it implies $\dot{A} < 0$. With $dS_{\rm mat} \approx 0$ this is in contradiction with (1). In the following we will illustrate this point and render it more precise.

We start in the next section by giving an example which demonstrates the validity of the GSL for "ordinary" matter. In section 3 we then study models with different equations of state. Finally, we summarize and conclude in section 4.

2 An example for the GSL at work

To gain some confidence in the generalized second law let us briefly review a simple example: a flat universe filled with radiation and a cosmological constant. For the metric

$$ds^2 = dt^2 - a(t)^2 \left(dr^2 + r^2 d\Omega^2\right) \ , \tag{2}$$

the proper distance to the future horizon is given by

$$D_{\rm FH}(t) = a(t) \int_t^\infty \frac{dt'}{a(t')} \ , \tag{3}$$

which leads to the horizon entropy (area)

$$S_{\rm FH} = 2\pi A_{\rm FH}(t) = 8\pi^2 D_{\rm FH}^2(t) \ . \tag{4}$$

The Hubble horizon is simply given by

$$D_{\rm Hub}(t) = \frac{1}{H(t)} \ , \tag{5}$$

and the corresponding entropy would be

$$S_{\rm Hub} = 2\pi A_{\rm Hub}(t) = 8\pi^2 \frac{1}{H^2(t)} \ . \tag{6}$$

The evolution of the scale factor $a(t)$ is determined by the Friedmann equations

$$3H^2 = \rho \ , \quad \dot{\rho} + 3H(\rho + p) = 0 \ . \tag{7}$$

In our simple example the energy density is the sum $\rho = \rho_{\rm R} + \rho_\Lambda$. Here, radiation and cosmological constant correspond to perfect fluids with the equations of state

$$p_{\rm R} = w_{\rm R}\rho_{\rm R} = \frac{1}{3}\rho_{\rm R} \ , \quad \text{and} \quad \mathrm{p}_\Lambda = \mathrm{w}_\Lambda \rho_\Lambda = -\rho_\Lambda = -\Lambda \ , \tag{8}$$

for the radiation and the cosmological constant component, respectively. For radiation (photons), energy and entropy density are determined by the temperature,

$$\rho_{\rm R} = \sigma T^4 \ , \quad s_{\rm R} = \frac{4}{3}\sigma T^3 \ , \quad \sigma = \frac{\pi^2}{15} \ . \tag{9}$$

[4]We will see that this is a reasonable assumption for the late universe, but violating this assumption is also one possibility to get a phantom cosmology in agreement with Eq. (1) (cf. [22]).

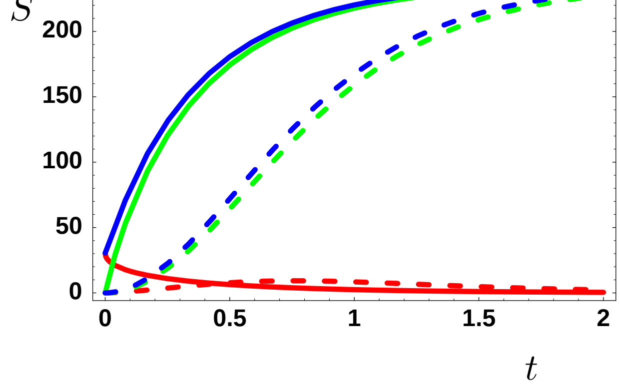


Figure 1: Time evolution of the entropy for a universe filled with radiation and a cosmological constant ($\Lambda = 1$). Matter entropy inside the horizon (red) and horizon entropy (green) add up to the total entropy (blue). Solid lines are for the future horizon, dashed lines for the Hubble horizon.

The total entropy inside the horizon[5] is then given by

$$S_{\rm R} = \frac{4\pi}{3} D_{\rm H}^3(t) s_{\rm R}(t) = \frac{16\pi}{9} \sigma T_0^3 \frac{D_{\rm H}^3(t)}{a^3(t)} \ . \tag{10}$$

For the last equality we have used that for a gas of massless particles $T = \frac{T_0}{a(t)}$, with $T_0 = T(t_0)$ and $a(t_0) \equiv 1$.

As shown in [21], the Friedmann equations can be solved analytically in this case, yielding

$$a(t) = \left(\frac{\sigma T_0^4}{\Lambda}\right)^{\frac{1}{4}} \left(\sinh(2\sqrt{\frac{\Lambda}{3}}\, t)\right)^{\frac{1}{2}} . \tag{11}$$

Inserting this result for the scale factor into Eqs. (4), (10) we obtain

$$\begin{aligned} S_{\rm FH} &= \frac{6\pi^2}{\Lambda} \sinh(x) J^2(x) \ , \\ S_{\rm R}^{\rm FH} &= \frac{2\pi}{\sqrt{3}} \sigma^{\frac{1}{4}} \Lambda^{-\frac{3}{4}} J^3(x) \ , \\ x = 2\sqrt{\frac{\Lambda}{3}} t, &\quad J(x) = \int_x^\infty \frac{dx}{(\sinh(x))^{\frac{1}{2}}} \ , \end{aligned} \tag{12}$$

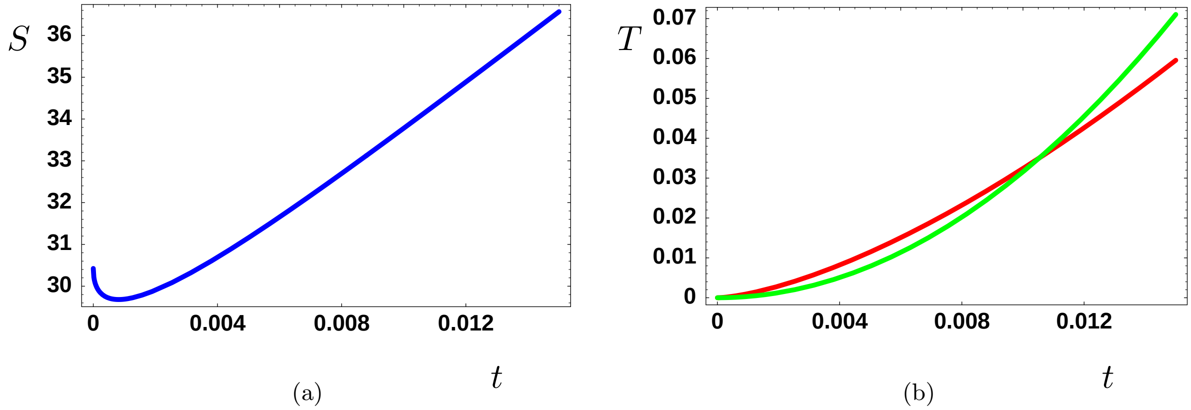
if we take the future horizon, and

$$\begin{aligned} S_{\rm Hub} &= \frac{24\pi^2}{\Lambda} (\tanh(x))^2 \ , \\ S_{\rm R}^{\rm Hub} &= \frac{16}{\sqrt{3}} \pi \sigma^{\frac{1}{4}} \Lambda^{-\frac{3}{4}} (\tanh(x))^{\frac{3}{2}} (\cosh(x))^{-\frac{3}{2}} \end{aligned} \tag{13}$$

for the Hubble horizon. We note that the entropies are independent of the initial temperature T_0 due to our choice of the initial time t_0. The results are plotted in Fig. 1.

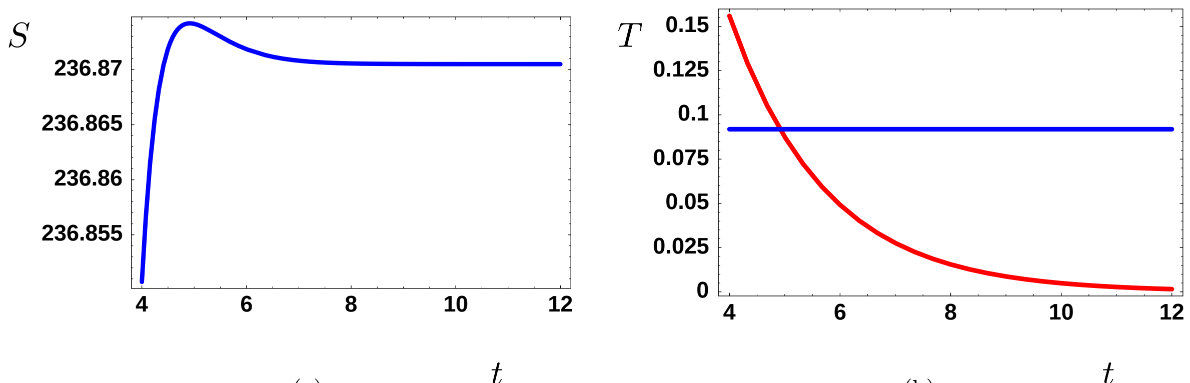
Independent of our choice for the horizon, i.e. for future horizon as well as Hubble horizon, the GSL, Eq. (1), appears to work. The blue curves, which represent the sum of matter entropy inside the horizon and horizon entropy, both seem to increase with time monotonously. However, with the help of a "magnifying glass" one can spot a small decrease in the total entropy at very early and very late times for the future horizon, Fig. 2(a) and Fig. 3(a). In case of the Hubble horizon there is no decrease at early times.

[5] We take a space-like volume and the equal-time is specified by an observer resting with respect to the fluid.



(a) (b)

Figure 2: Left panel, total entropy for very early times (parameters as in Fig. 1) using the future horizon. In the right panel the total matter entropy inside a Hubble volume (red) is compared to the maximal entropy allowed by the Bousso bound [23] (green).



(a) (b)

Figure 3: Left panel: evolution of the total entropy for future horizon at late times; for the Hubble horizon the picture is very similar (parameters as in Fig. 1). Right panel: comparison of the radiation temperature $T = T_0/a(t)$ (red) with the horizon temperature $1/(2\pi D_{\rm H}(t))$ (blue).

Consider first the small drop in the total entropy at early times, Fig. 2(a). We don't have to worry about this decrease for two reasons. First, one can easily check that at these early times the temperature is well above the Planck scale, so that we cannot expect the formula Eq. (10) for the matter entropy to hold. Second, the covariant entropy bound [23] is not fulfilled at these early times. This is depicted in Fig. 2(b) where the total matter entropy inside a Hubble volume is compared with the maximally allowed entropy inside this volume [23]. In the region where the entropy decreases the covariant entropy bound is clearly violated, and the used expression for the entropy is therefore inapplicable.

Let us now turn to the drop in entropy at very late times, Fig. 3(a)[6]. At the corresponding temperatures the typical wavelength of photons contributing to the matter entropy, $\lambda \sim 1/T$, is bigger that the horizon size at this time. Since the wavelength doesn't fit into the horizon anymore, the flat space relation Eq. (10) is no longer applicable. In Fig. 3(b) the radiation temperature $T = T_0/a(t)$ is compared to the "horizon temperature" $1/(2\pi D_{\rm H}(t))$, which can roughly be thought of as the smallest possible temperature in an expanding universe. Again, the drop in entropy occurs when Eq. (10) is no longer applicable.

In summary, we find that the GSL works independent of our choice of horizon (future horizon or Hubble horizon). Indeed, apparent violations of the GSL can be easily understood as consequence of the inapplicability of the flat space expression Eq. (10) for the radiation

[6]Note that the time between Planck scale and Hubble scale temperatures, where the GSL is valid, is rather small due to the large value chosen for the cosmological constant, $\Lambda = 1$ in Planck units.

entropy.

3 Decreasing horizon entropy in phantom models

Having gained some confidence in the GSL by studying the example in the previous section, let us now move on to a more interesting situation. Consider the following simple model for the equation of state,

$$w(\rho_{\rm DE}) = -1 + \Delta w\, \Theta(\rho_{\rm max} - \rho_{\rm DE}). \tag{14}$$

For $\Delta w < 0$, the dark energy has phantom behaviour at early times. ¿From the Friedmann equations one finds that the energy density $\rho_{\rm DE}$ grows until it reaches the maximal density $\rho_{\rm max}$ which acts like a cosmological constant. In this way the "big rip" singularity is avoided which occurs for a constant equation of state $w(\rho_{\rm DE}) < -1$. $\Delta w = 0$ is the case of a cosmological constant. For $\Delta w > 0$ one obtains a quintessence-like model with constant equation of state at late times. At early times the maximal density $\rho_{\rm max}$ is not exceeded, avoiding the initial singularity. For $\Delta w < \frac{2}{3}$ one has an accelerated expansion of the universe which is associated with the formation of a future horizon.

In Fig. 4 the evolution of the total entropy $S = S_{\rm rad} + S_{\rm H}$ is shown for a quintessence model, a cosmological constant and a phantom model, respectively. In addition to the dark energy, described by the equation of state given in Eq. (14), we have added a component of dark matter ($w = 0$, $S_{\rm DM} = 0$) and a radiation component ($w = \frac{1}{3}$). η is the conformal time ($d\eta = dt/a(t)$), with $\eta(t_0) = 0$. The energy densities are fixed by their current values, $T_{\rm rad} = 2.7K$, $H_0 = 70$ km/(s Mpc) $\sim 6 \times 10^{-61}$ $\rm M_P$ and $\Omega_{\rm DE}/\Omega_{\rm DM} = 0.7/0.3$, where Ω_{DE} and $\Omega_{\rm DM}$ are the fractions of the total energy density contributed by dark energy and dark matter, respectively. The total entropies of the three models are plotted in Fig. 4(a). The entropy increases both for quintessence and for the cosmological constant. In the quintessence model the entropy increases without bound, as the horizon continues to grow with time. In both models the GSL is fulfilled. On the contrary, for the phantom model the total entropy first increases but then decreases in violation of the GSL. In particular today, at $\eta = 0$, the phantom model is inconsistent with the GSL.

In Fig. 4(b) the horizon entropy for the phantom model is compared with the radiation entropy inside the horizon. Most of the time the horizon entropy is larger than the matter entropy by many orders of magnitude. One easily verifies that this holds for all three models. Moreover, the matter entropy inside the horizon decreases with time. In fact, one can infer from the Friedmann equations that accelerated expansion leads to a decrease in entropy inside the horizon for all components which expand adiabatically such as radiation. Hence, as long as we consider only adiabatic expansion where no matter entropy is generated for any component, the GSL is always violated as soon as the horizon and therefore the horizon entropy begins to shrink.

What can we learn from this? If $\rho_{\rm max}$ is sufficiently large, $\rho_{\rm max} > \rho_0$, where ρ_0 is the current total energy density of the universe, then the GSL is violated for all $\Delta w < 0$. The more conservative requirement $\rho_{\rm max} = \rho_0^{\rm DE}$, where $\rho_0^{\rm DE}$ is the current dark energy density, would allow $\Delta w < 0$ only in the past, with a cosmological constant in the future making the current era very special. In our simple model one then finds that consistency with the GSL requires $\Delta w > -1.8$ if we use the future horizon and $\Delta w > -0.43$ for the Hubble horizon.

4 Summary and Conclusions

Generically, gravity leads to the formation of horizons which separate causally disconnected regions. The second law of thermodynamics, which applies to closed systems, then has to be modified. The conjecture of a generalized second law (GSL), which applies to the sum of

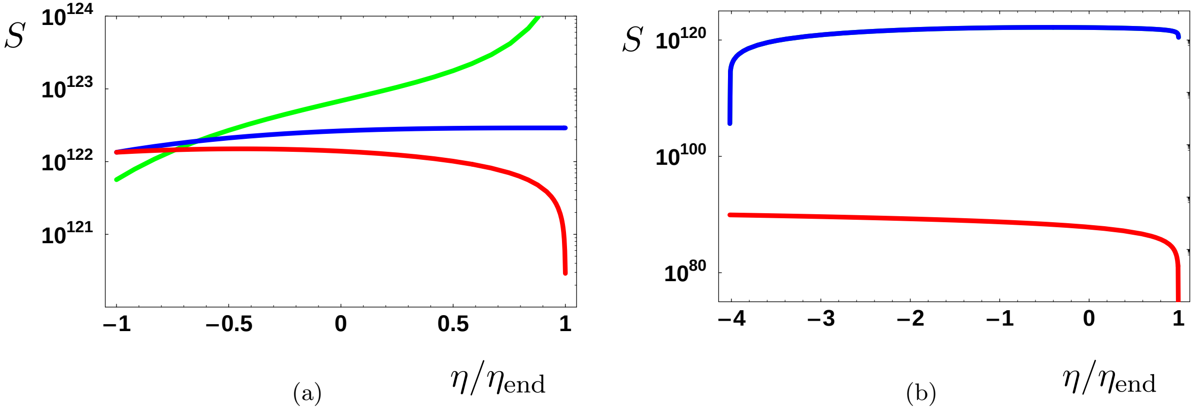


Figure 4: Left panel: evolution of the total entropy $S = S_{\rm rad} + S_{\rm H}$ with (conformal) time η (rescaled by $\eta_{\rm end}$, the conformal time at $t = \infty$). Parameters: $\Omega_{\rm mat} = 0.3$, $\Omega_{\rm DE} = 0.7$, $T_{\rm rad} = 2.7K$. The green curve represents a quintessence universe ($\Delta w = 0.25$), the blue one a cosmological constant ($\Delta w = 0$) and the red one a phantom universe ($\Delta w = -0.25$); $\rho_{\rm max} = 100\rho_{\rm today}$. Right panel: comparison of horizon entropy (blue) and radiation entropy (red) for the phantom model. The radiation entropy is negligible at late times. In all cases the future horizon has been chosen; the results for the Hubble horizon are qualitatively similar.

ordinary matter entropy and horizon entropy, is well established for black holes which represent stationary systems. For non-stationary systems, as they appear in cosmology, the status of the GSL is much more speculative and several questions, like the proper choice of the horizon, remain to be clarified. Yet, as demonstrated by our simple example of a universe made of radiation and a cosmological constant, the GSL appears to work in cosmological situations, too.

Keeping the above caveats in mind it is nevertheless interesting to apply the GSL and ask what it can tell us about the dark energy equation of state. In general, both quintessence, $w_{\rm DE} > -1$, and a cosmological constant, $w_{\rm DE} = -1$, are consistent with a GSL. On the contrary, long phases of a phantom equation of state, $w_{\rm DE} < -1$, typically lead to a decrease of the total entropy. Short phases with a phantom equation of state might be allowed.

In conclusion, further studies on the validity of a generalized second law of thermodynamics are an important theoretical challenge. Our simple example of the dark energy equation of state already illustrates its potential as a tool for cosmology.

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References

[1] A. G. Riess *et al.* [Supernova Search Team Collaboration], Astrophys. J. **607**, 665 (2004) [astro-ph/0402512].

[2] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003) [astro-ph/0302209].

[3] M. Tegmark *et al.* [SDSS Collaboration], Phys. Rev. D **69** (2004) 103501 [astro-ph/0310723].

[4] Ya. B. Zeldovich, Pis'ma Zh. Eksp. Teor. Fiz. **6** (1967) 883 [JETP Lett. **6** (1967) 316].

[5] S. Weinberg, Rev. Mod. Phys. **61** (1989) 1.

[6] C. Wetterich, Nucl. Phys. B **302**, 668 (1988)

[7] B. Ratra and P. J. Peebles, Phys. Rev. D **37**, 3406 (1988)

[8] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998) [astro-ph/9708069].

[9] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91** (2003) 071301 [astro-ph/0302506].

[10] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002) [astro-ph/9908168].

[11] S. M. Carroll, M. Hoffman and M. Trodden, Phys. Rev. D **68** (2003) 023509 [astro-ph/0301273].

[12] J. D. Bekenstein, Phys. Rev. D **7** (1973) 2333.

[13] J. D. Bekenstein, Phys. Rev. D **9** (1974) 3292.

[14] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15** (1977) 2738.

[15] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15** (1977) 2752.

[16] W. G. Unruh and R. M. Wald, Phys. Rev. D **25** (1982) 942.

[17] P. C. W. Davies, Class. Quant. Grav. **4** (1987) L225.

[18] P. C. W. Davies, Class. Quant. Grav. **5** (1988) 1349.

[19] For a recent discussion, see: L. Susskind and J. Lindesay, "An introduction to black holes, information and the string theory revolution", World Scientific, Singapore, 2005

[20] T. M. Davis, P. C. W. Davies and C. H. Lineweaver, Class. Quant. Grav. **20** (2003) 2753 [astro-ph/0305121].

[21] P. C. W. Davies and T. M. Davis, Found. Phys. **32 (12)** (2002) 1877 [astro-ph/0310522].

[22] W. Buchmüller and J. Jaeckel, in preparation.

[23] R. Bousso, JHEP **9907** (1999) 004 [hep-th/9905177].

[24] R. Brustein, Phys. Rev. Lett. **84** (2000) 2072 [gr-qc/9904061].

[25] G. Izquierdo and D. Pavon, Phys. Rev. D **70** (2004) 127505 [gr-qc/0501039].