Self-dual and non self-dual axially symmetric $SU(2)$ calorons

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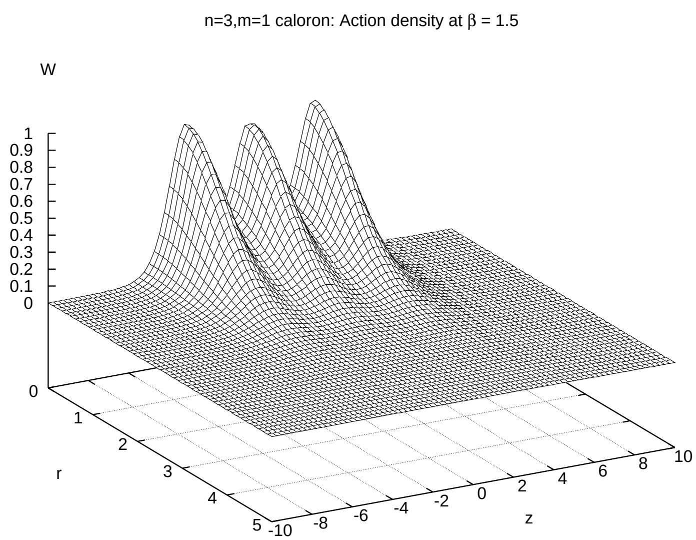
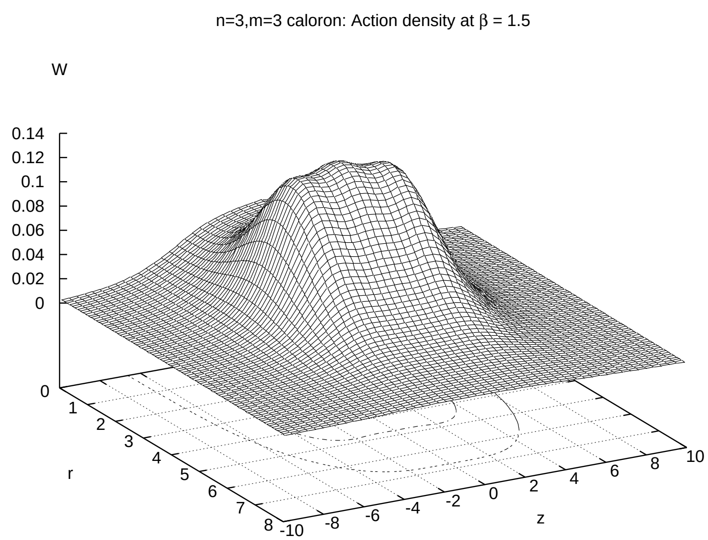




Quarks International Seminar, Repino, 23 May, 2006

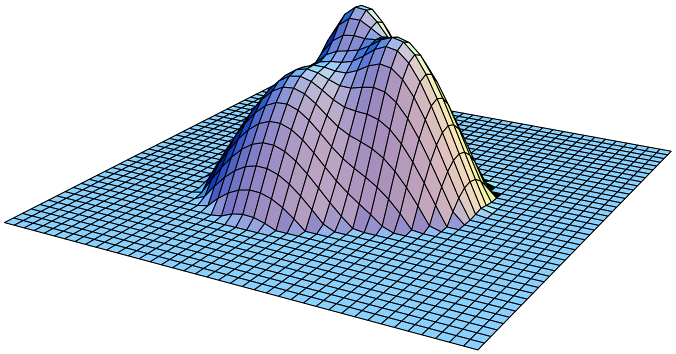
<u>Caloron solution:</u> $SU(2)$ instanton at finite temperature for which A_0 approaches a constant $\beta(r^a \cdot \sigma^a)$ as $r \to \infty$. (*Harrington & Shepard (1978), Nahm (1984), Lee & Yi (1997), van Baal & Kraan (1998)...*)

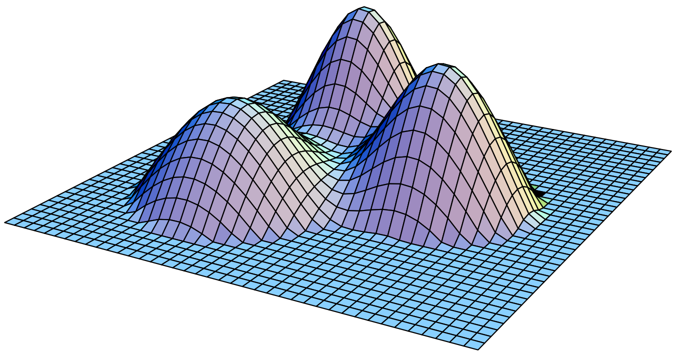
Non-trivial Polyakov loop (holonomy) around S^1 at spacial infinity

$$\mathcal{P}(\mathbf{r}) = P\exp\left(\int_0^T A_0(\mathbf{r}, x_0)dx_0\right), \quad T = \frac{2\pi}{\beta}$$

The period T is the inverse temperature of eucledial field theory. Non-trivial value of the Polyakov loop modify the vacuum action density and reveal constitutent structure of caloron.

Self-dual solution: Charge one periodic $SU(2)$ instanton is build out of monopoles twisted by $\mathcal{P}$. The BPS constituents are not interacting but the positions of the monopoles depend on β.
Van Baal – Kraan solution (Phys.Lett., B435, 1998):





I. Axially symmetric Ansatz for the $SU(2)$ euclidean Yang-Mills model

$$A_\mu dx^\mu = \left(\frac{K_1}{r}dr + (1-K_2)d\theta\right)\frac{\tau_\varphi^{(n)}}{2e} - n\sin\theta\left(K_3\frac{\tau_r^{(n,m)}}{2e} + (1-K_4)\frac{\tau_\theta^{(n,m)}}{2e}\right)d\varphi;$$

$$A_0 = A_0^a\frac{\tau^a}{2} = \eta\left(K_5\frac{\tau_r^{(n,m)}}{2} + K_6\frac{\tau_\theta^{(n,m)}}{2}\right) \; .$$

We define the triplet of spacial unit vectors

$$\hat{e}_r^{(n,m)} = [\sin(m\theta)\cos(n\varphi), \sin(m\theta)\sin(n\varphi), \cos(m\theta)];$$

$$\hat{e}_\theta^{(n,m)} = [\cos(m\theta)\cos(n\varphi), \cos(m\theta)\sin(n\varphi), -\sin(m\theta)];$$

$$\hat{e}_\varphi^{(n)} = [-\sin(n\varphi), \cos(n\varphi), 0]$$

and the $su(2)$ basis

$$\tau_r^{(n,m)} = \tau^a\cdot\hat{e}_r^{(n,m)} = \sin(m\theta)\tau_\rho^{(n)} + \cos(m\theta)\tau_z = \begin{pmatrix} \cos m\theta & \sin m\theta\; e^{-in\varphi} \\ \sin m\theta\; e^{in\varphi} & -\cos m\theta \end{pmatrix};$$

$$\tau_\theta^{(n,m)} = \tau^a\cdot\hat{e}_\theta^{(n,m)} = \cos(m\theta)\tau_\rho^{(n)} - \sin(m\theta)\tau_z = \begin{pmatrix} -\sin m\theta & \cos m\theta\; e^{-in\varphi} \\ \cos m\theta\; e^{in\varphi} & \sin m\theta \end{pmatrix};$$

$$\tau_\varphi^{(n)} = \tau^a\cdot\hat{e}_\varphi^{(n)} = -\sin(n\varphi)\tau_x + \cos(n\varphi)\tau_y = \begin{pmatrix} 0 & -ie^{-im\varphi} \\ ie^{im\varphi} & 0 \end{pmatrix},$$

where $\tau_\rho^{(n)} = \cos(n\varphi)\tau_x + \sin(n\varphi)\tau_y$, $\rho = \sqrt{x^2+y^2} = r\sin\theta$

Some properties of the anzatz:

Axial symmetry: Rotations around z-axis can be compensated by an abelian gauge transformation $U = \exp\left\{\frac{i}{2}\tau_\varphi^{(n)}\omega(r,\theta)\right\}$

Boundary conditions:

Regularity at the origin ($r \to 0$):

$$K_1(0,\theta) = 0 \ , \quad K_2(0,\theta) = 1 \ , \quad K_3(0,\theta) = 0 \ , \quad K_4(0,\theta) = 1 \ ,$$

$$\sin(m\theta)K_5(0,\theta) + \cos(m\theta)K_6(0,\theta) = 0 \qquad \partial_r \left[\cos(m\theta)K_5(r,\theta) - \sin(m\theta)K_6(r,\theta)\right]\big|_{r=0} = 0.$$

$r \to \infty$:

$$K_1 \longrightarrow 0 \ , \quad K_2 \longrightarrow 1 - m \ , \quad K_3 \longrightarrow \frac{\cos\theta - \cos(m\theta)}{\sin\theta} \ \ (\text{for odd } m) \ ,$$

$$K_3 \longrightarrow \frac{1 - \cos(m\theta)}{\sin\theta} \ \ (\text{for even } m) \ , \quad K_4 \longrightarrow 1 - \frac{\sin(m\theta)}{\sin\theta} \ , \quad K_5 \longrightarrow 1 \ , \quad K_6 \longrightarrow 0 \ .$$

Regularity on the z axis ($\theta \to 0, \pi$):

$$K_1 = K_3 = K_6 = 0 \ , \quad \partial_\theta K_2 = \partial_\theta K_4 = \partial_\theta K_5 = 0 \ ,$$

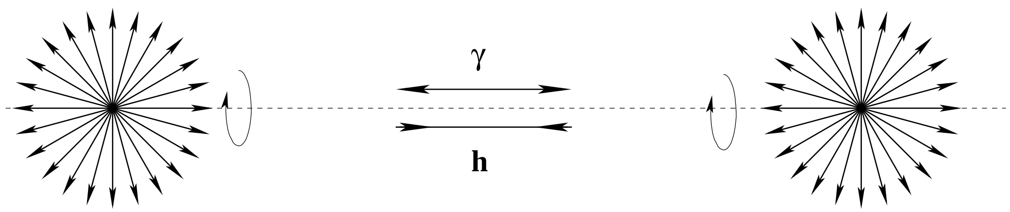
Asymptotic behaviour:

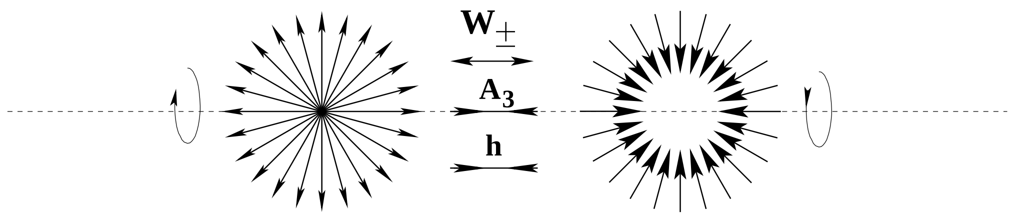
$$\text{for even } m = 2k \ : \ A_0 \longrightarrow \beta \hat{e}_r^{(n,m)} = \beta U \tau_z U^\dagger \ , \quad A_k \longrightarrow i\partial_k U U^\dagger \ ,$$

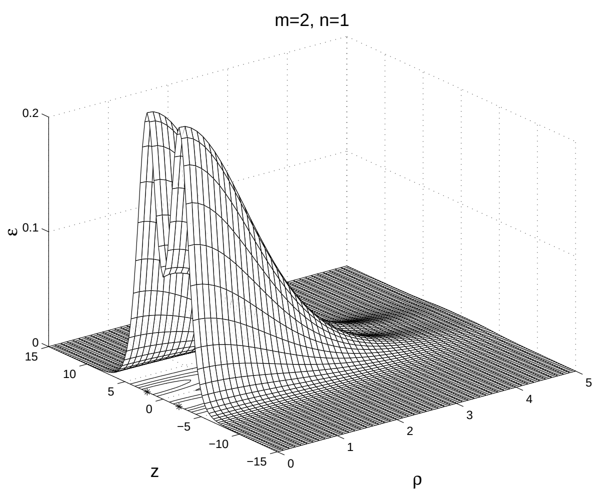
$$\text{for odd } m = 2k+1 \ : \ A_0 \longrightarrow \beta \hat{e}_r^{(n,m)} \ , \quad A_\mu \longrightarrow U A_{\mu\infty}^{(n,1)} U^\dagger + i\partial_\mu U U^\dagger \ ,$$

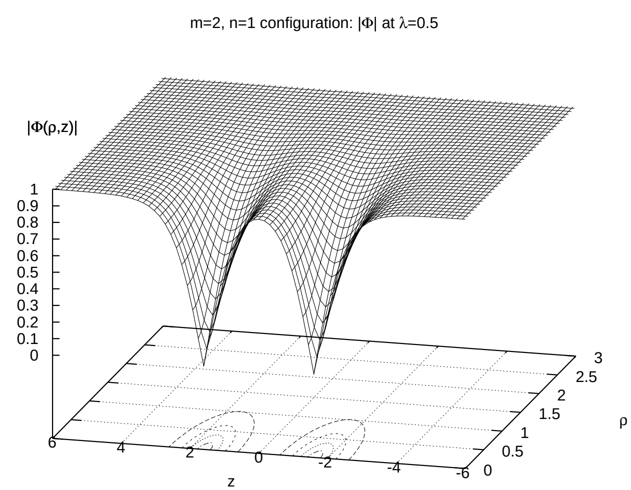
where $U = \exp\{-ik\theta\tau_\varphi^{(n)}\}$ and $A_{\mu\infty}^{(n,1)}$ is the self-dual charge n instanton solution.

How to construct a magnetic dipole?









Self-dual caloron vs non-self dual caloron

Self-dual caloron:

The Eucledian action

$$S = \frac{1}{2e^2}\int\limits_0^T d\tau \int d^3x \,\mathrm{Tr}\, F_{\mu\nu}^2 = \frac{1}{16\pi^2}\int\limits_0^T d\tau \int d^3x \,\mathrm{Tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu} = \frac{8\pi^2}{e^2}N$$

The first order self-duality equations $\widetilde{F}_{\mu\nu} = \pm F_{\mu\nu}$ yield the absolute munimum of the action.

(i) $S = 8\pi^2/e^2 Q$;

(ii) No net interaction between self-dual constitutents; Coulomb electromagnetic repulsion is compencated due to long-range scalar interaction.

non-self dual calorons:

(i) They are solutions of second order Yang-Mills equations $\partial_\mu F_{\mu\nu} = 0$;

(ii) $S > 0$, even if $N = 0$ (deformations of the trivial sector);

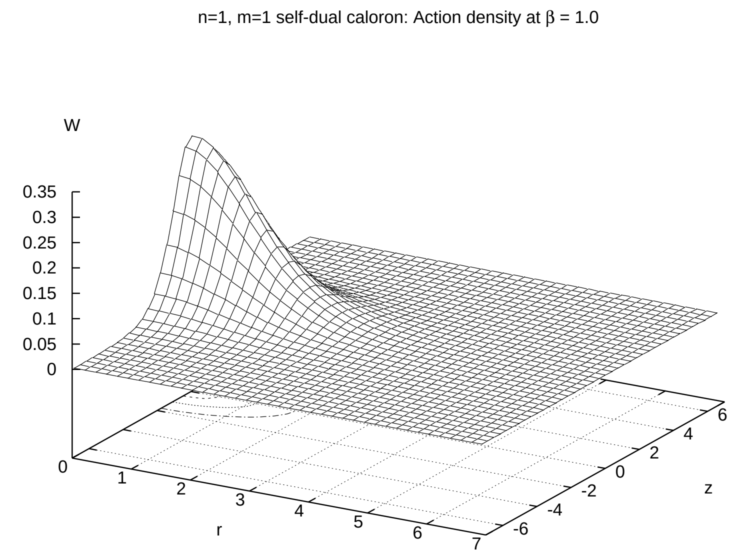
(iii) The consituents are non-BPS monopoles and/or vortices which are in static equilibrium; no long range forces but the separation is small.

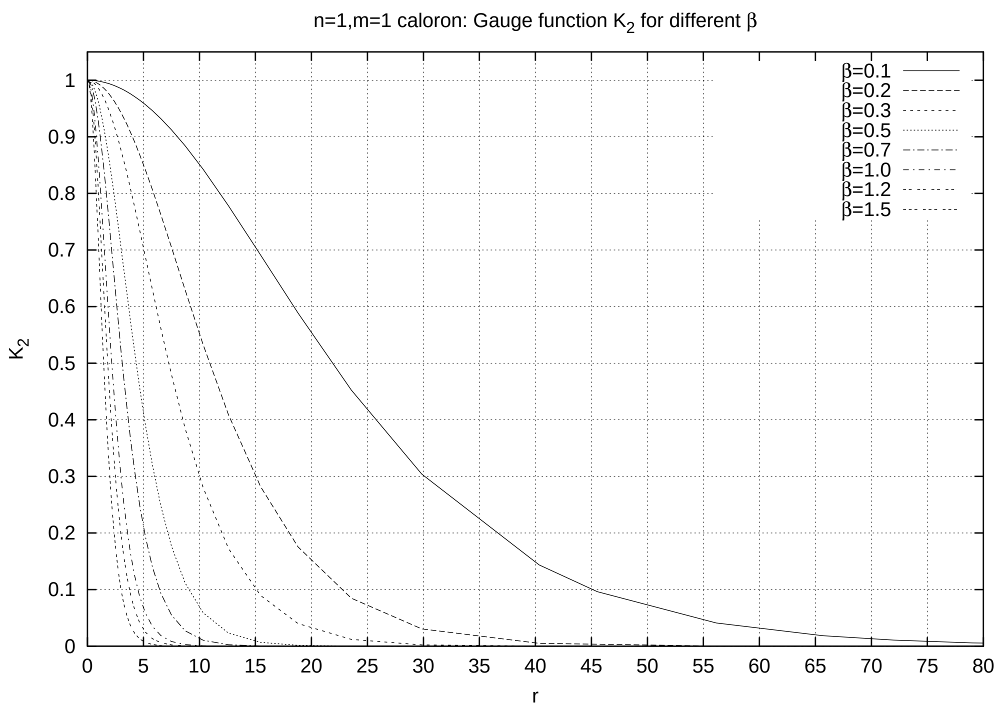
II. Numerical results

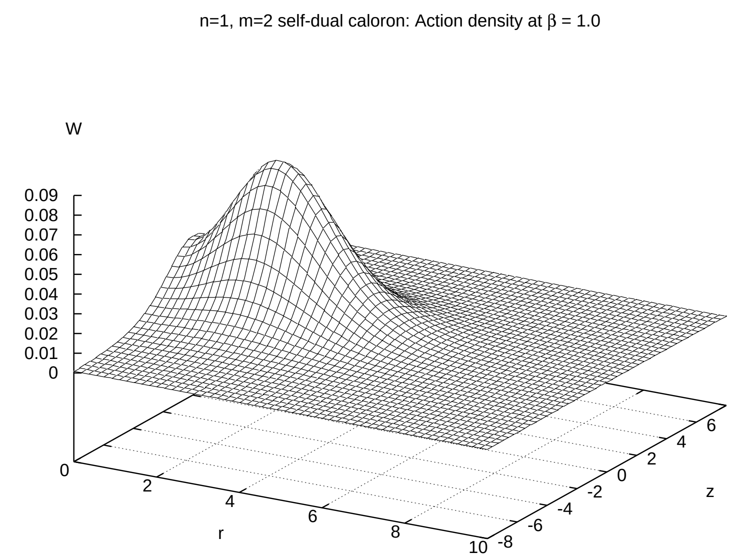
m=1, $n = 1, 2, 3\ldots$ (Self-dual calorons)

Topological charge

$$N = \frac{1}{16\pi^2}\int_0^T d\tau \int d^3x \,\mathrm{Tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu} = \frac{1}{2} n \left(1-(-1)^m\right)$$

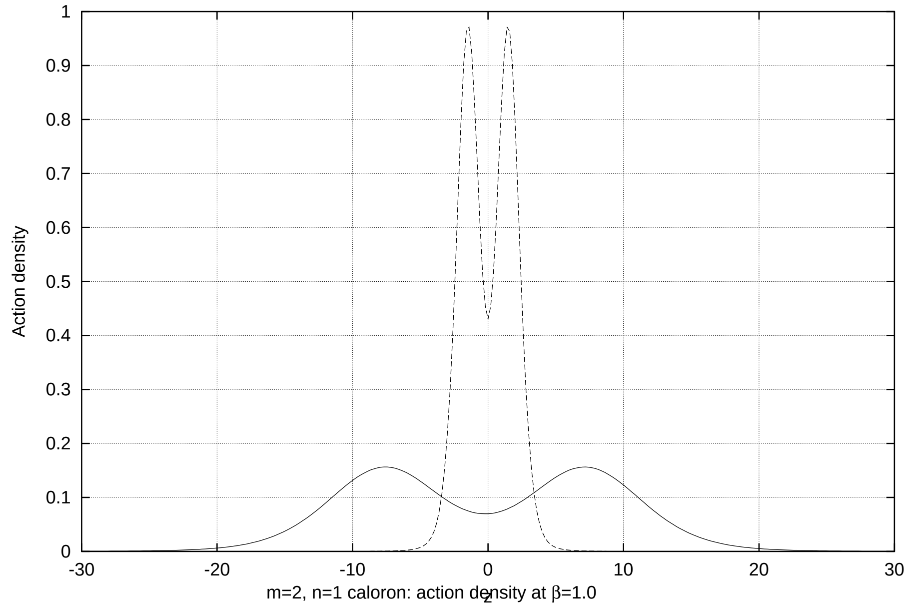


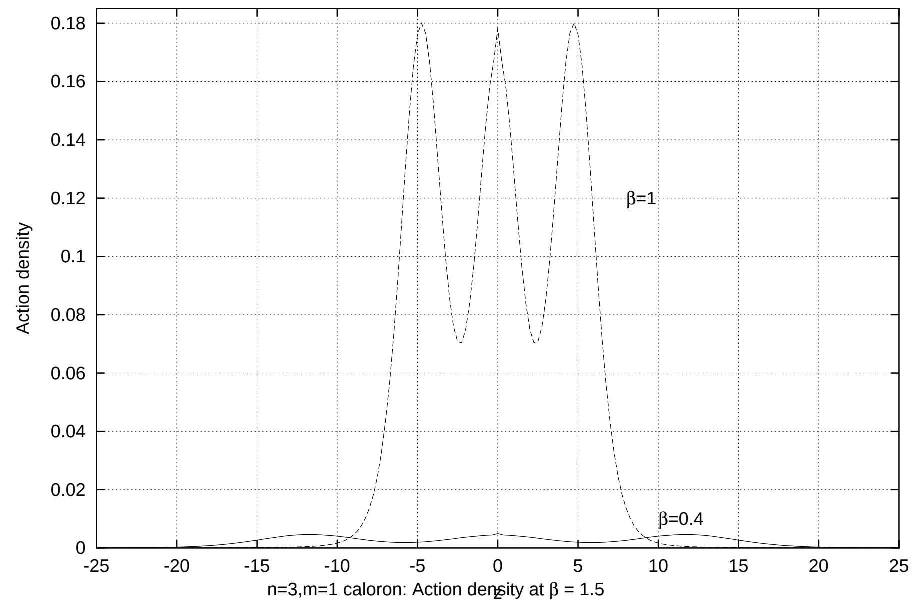


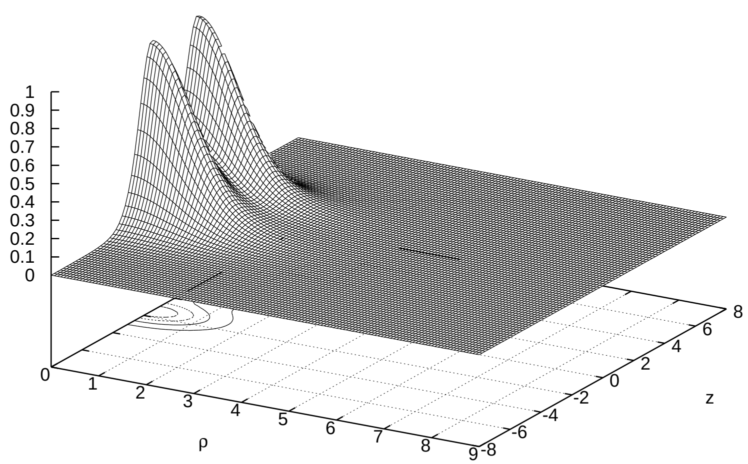


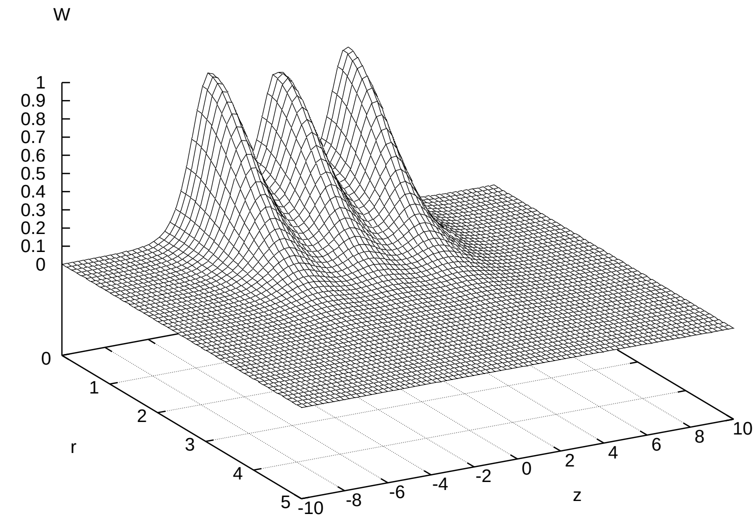
Non self-dual calorons

n=1,2 m=2,3, . . . (Monopole-antimonopole chains)

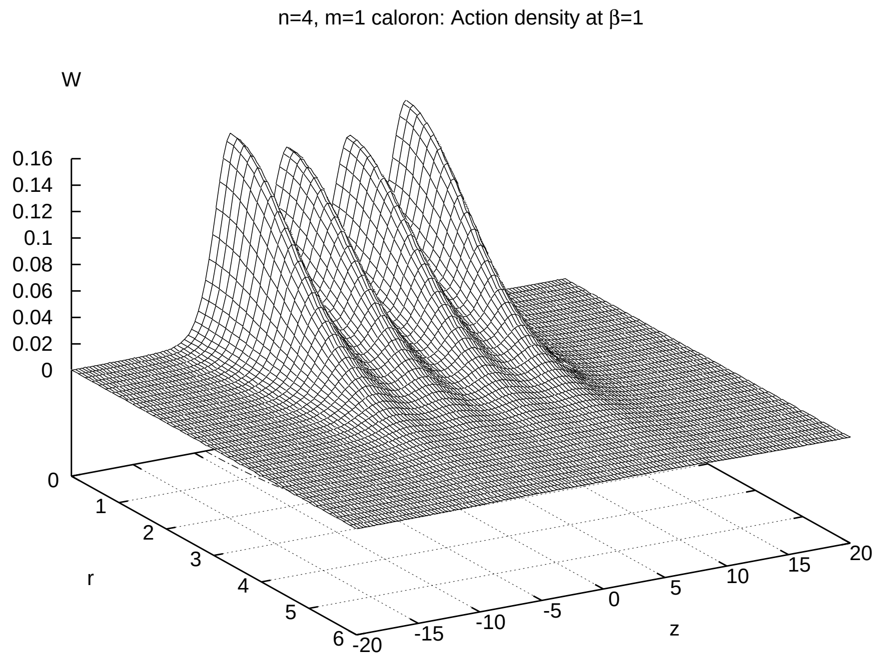


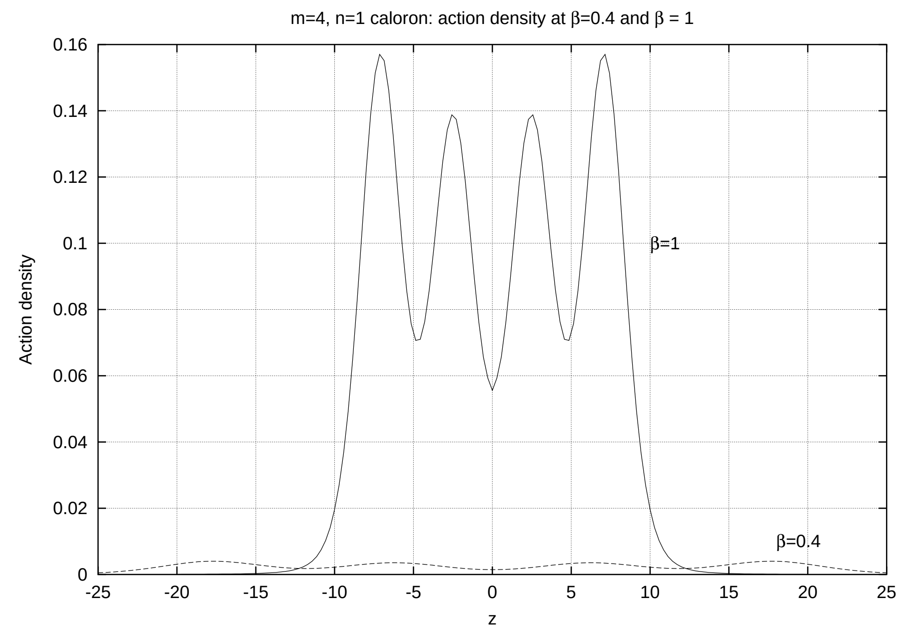






M-A-M-A chain $(n = 1, m = 4)$

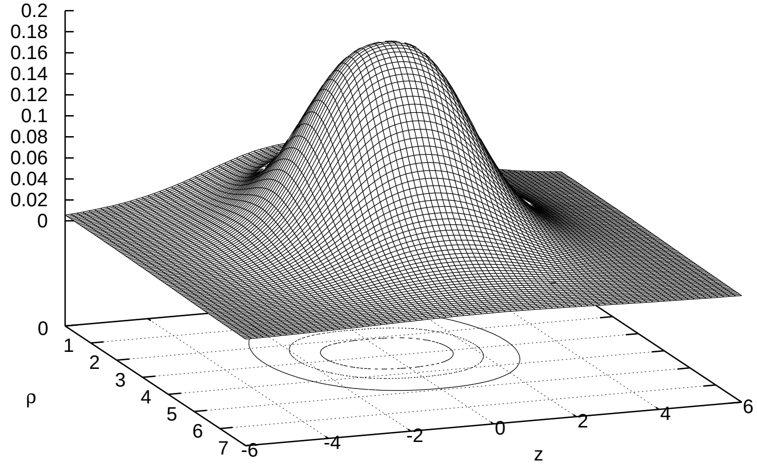




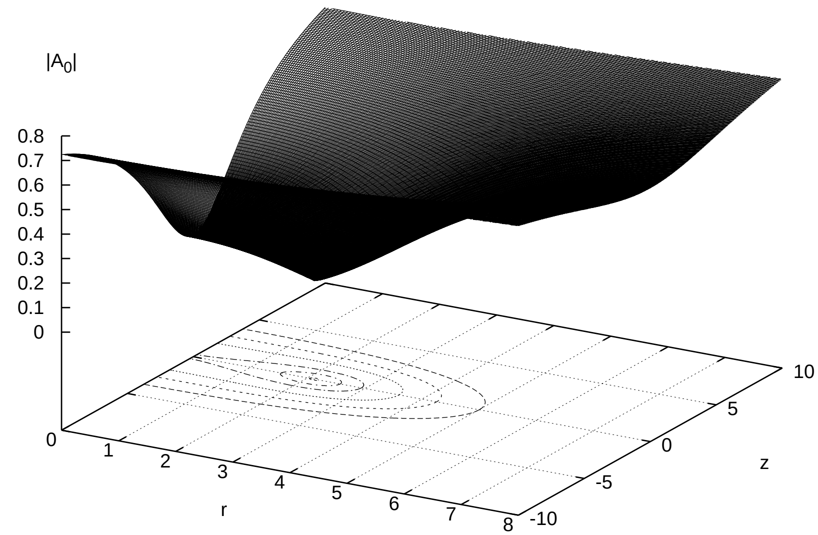
These non-self dual configurations exist as saddlepoints of the action functional (Non-contractible monopole loops – *Taubes 1981*)

n=2,3,... m=3,4, ... (Vortex-like caloron structures)

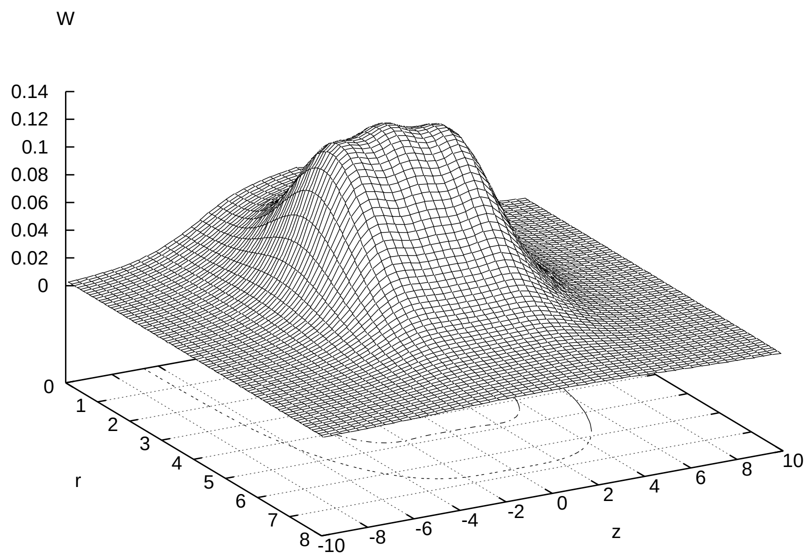
m=2, n=3 caloron: action density at β=1.0



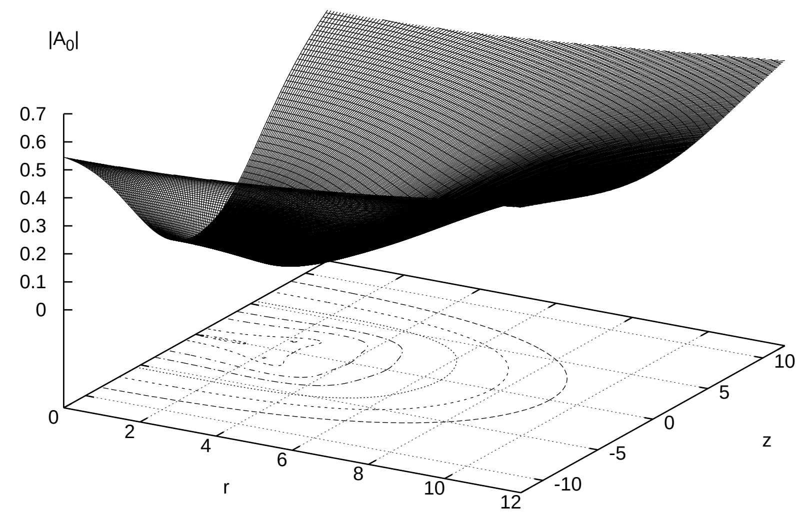
m=2, n=3 caloron: $|A_0|$ at β=1



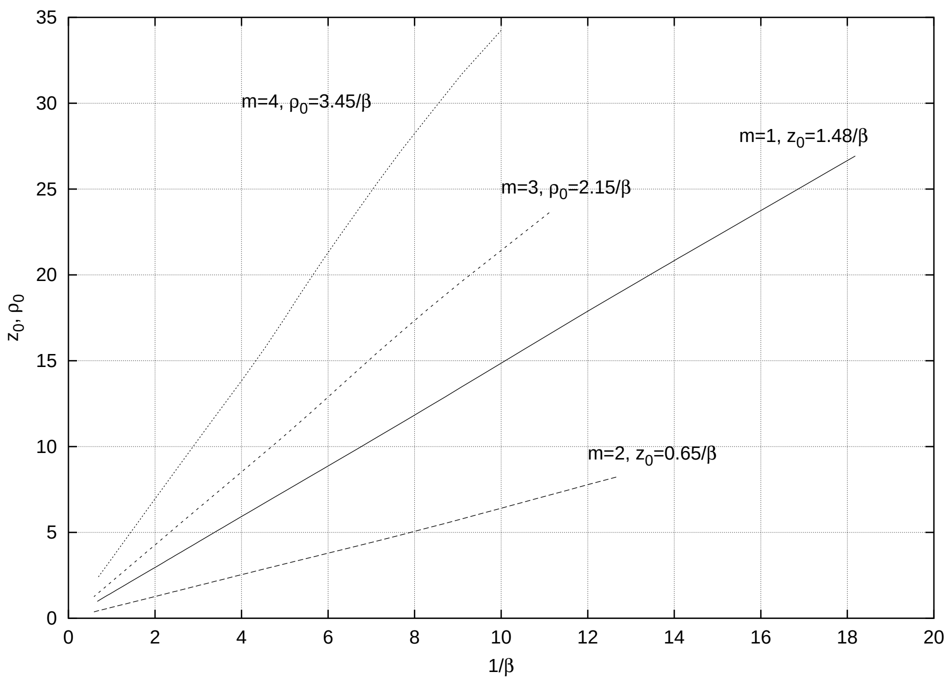
n=3,m=3 caloron: Action density at β = 1.5

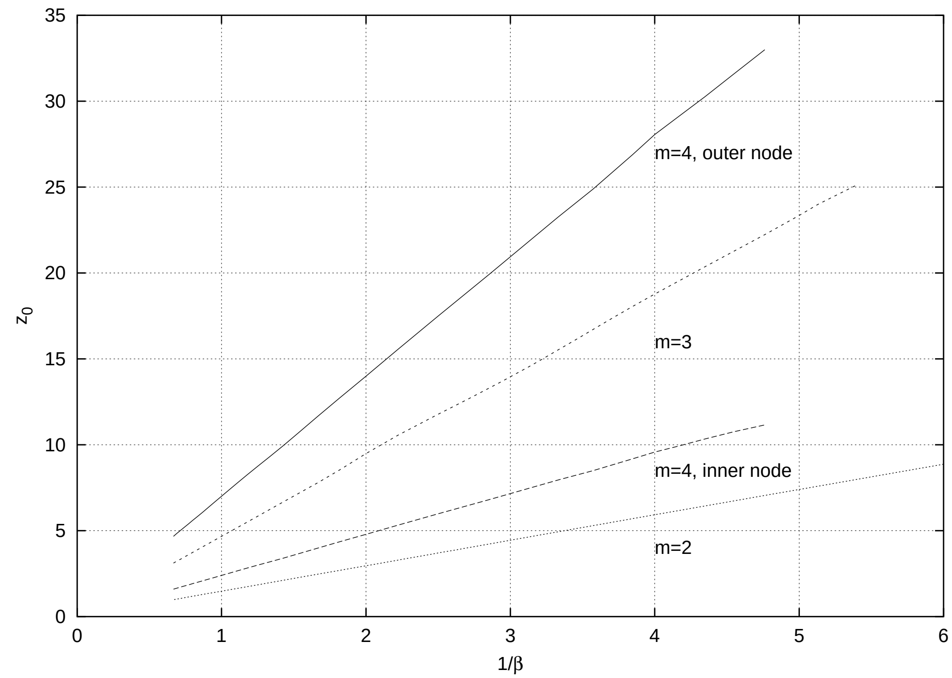


m=3, n=3 caloron: $|A_0|$ at β=1



Position of the constituent monopoles as function of temperature





The maxima of the action density, associated with positions of the constituent monopoles, or closed vortices, linearly depend on $1/\beta$.

Summary and outlook

- *New static axually symmetric* $SU(2)$ *non-self dual caloron solutions are presented;*
- *The constitutents of the solutions are monopole-antimonopole chains and circular vortices;*
- *Position of the constituent monopoles linearly depends on* $1/\beta$*;*
- *Relation with knotted solutions (á la Faddev-Niemi)?*
- *Further extensions of the model, e.g., interaction with fermions and consideration of* $SU(3)$ *calorons (work in progress).*