Hadronic effects in the muon anomalous magnetic moment

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Abstract

We discuss hadronic light-by-light scattering contribution to the muon anomalous magnetic moment, $a_\mu = (g_\mu - 2)/2$ paying particular attention to the consistent matching between the short- and the long-distance behavior of the light-by-light scattering amplitude. Accounting for the short-distance constraints leads to approximately 50% increase in the central value of $a_\mu^{\rm LbL}$, compared to existing estimates.

1 Introduction

In this talk I discuss theory of hadronic effects in the muon anomalous magnetic moment. The presentation is based on works [1, 2, 3] . The outline of the talk is [1]

- Decomposition of the muon anomalous magnetic moment
- Hadrons in polarization operator
- Hadrons in light-by-light
- Summary

Let me start from reminding a definition of the magnetic moment via the energy in the external magnetic field $\vec{B}$,

$$E = -\vec{\mu}\vec{B}\,, \quad \vec{\mu} = g_\mu \,\frac{e\hbar}{2m_\mu c}\,\vec{s}\,. \tag{1}$$

[1] I omit discussion on hadronic effects in the electroweak corrections.

From the Dirac equation $g_\mu = 2$. Deviations are due to radiative corrections,

$$g_\mu = 2\Big(1 + \frac{\alpha}{2\pi} + \dots\Big)\,, \qquad \text{Schwinger '48.} \tag{2}$$

The anomalous magnetic moment of muon is measured with a very high precision in the E821 experiment at BNL [4, 5],

$$\begin{aligned} a_{\mu^+}^{\exp} &= \frac{g_{\mu^+} - 2}{2} = 116\,592\,030(80) \times 10^{-11} \qquad \text{'02,} \\ a_{\mu^-}^{\exp} &= \frac{g_{\mu^-} - 2}{2} = 116\,592\,140(80) \times 10^{-11} \qquad \text{'04.} \end{aligned} \tag{3}$$

Assuming CPT invariance the average is also given

$$a_\mu^{\exp} = 116\,592\,080(60) \times 10^{-11}\,. \tag{4}$$

These experimental results are presented together with the theoretical predictions in Fig. 1.

The Standard Model prediction for a_μ can be represented as a sum

$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm had} + a_\mu^{\rm EW}\,. \tag{5}$$

The QED part involving only leptons and photons is the main one [6],

$$a_\mu^{\rm QED} = 116\,584\,720.7(1.2) \times 10^{-11}\,. \tag{6}$$

This accounts for one-, two-, three- and four-loop contributions, i.e., up to the α^4 terms.

Next is the hadronic contribution.

$$a_\mu^{\rm had} = a_\mu^{\rm had,LO} + a_\mu^{\rm had,HO} + a_\mu^{\rm LbL}\,. \tag{7}$$

The leading order hadronic contribution is diagrammatically represented in Fig. 2 by the quark loop while the the diagram in Fig. 3 present an example of the higher order hadronic contributions,

$$a_\mu^{\rm had,LO} = \begin{cases} 6963(62)(36)\times 10^{-11} & e^+e^- \text{ based,} \\ 7110(50)(8)(28)\times 10^{-11} & \tau \text{ based.} \end{cases} \tag{8}$$

The estimate [7] for the higher order term is

$$a_\mu^{\rm h,HO} = -100(6)\times 10^{-11}\,, \tag{9}$$

while for the light-by-light contribution [7]

$$a_\mu^{\rm LbL} = 86(35) \times 10^{-11}\,. \tag{10}$$

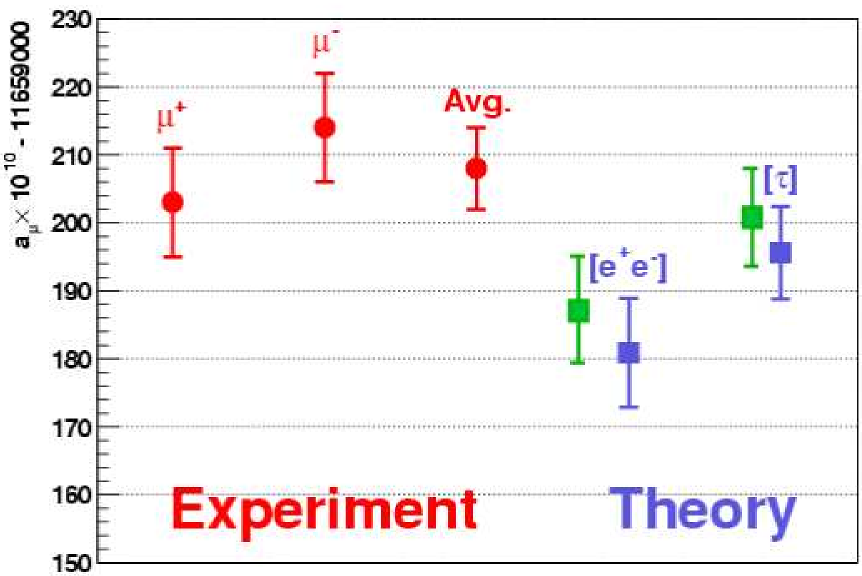

Figure 1: Experimental values and theoretical predictions. The green bars are due to the shift in the hadronic light-by-light contribution.

2 Hadrons in polarization operator

In theory

$$a_\mu^{\rm had,LO} = I = \left(\frac{\alpha\, m_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^{\infty} \frac{{\rm d}s}{s^2}\, K(s) R(s) \tag{11}$$

where $K(s)$ is the known function, $K(s) \to 1$ at $s \gg m_\mu^2$ and $R(s)$ is the cross section of e^+e^- annihilation into hadrons in units of $\sigma(e^+e^- \to \mu^+\mu^-)$. In the integration over s two regions can be single out. The threshold region $s \sim 4m_\pi^2$ where

$$R(s) \approx \frac{1}{4}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2}, \tag{12}$$

and the resonance region $s \sim m_\rho^2$ where by quark-hadron duality

$$R(s) \approx N_c \sum Q_q^2 \,. \tag{13}$$

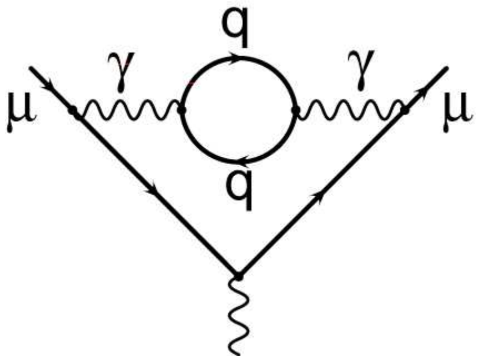

Figure 2: Lowest order hadronic contribution represented by a quark loop

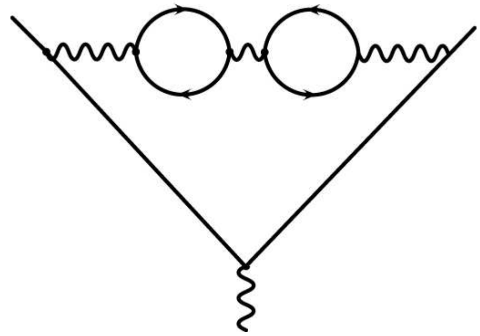

Figure 3: An example of higher order hadronic contribution

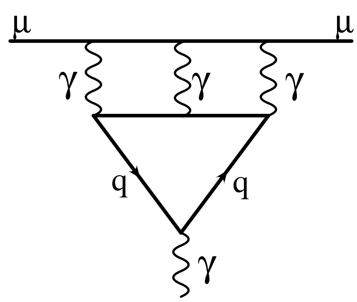

Figure 4: Light-by-light scattering contribution

In Fig. 4 the three-loop diagram is shown for the hadronic light-by-light contribution. This diagram includes the quark loop for the light-by-light scattering.

The leading order hadronic contribution $a_\mu^{\rm had,LO}$ is defined by experimental data from two sources; e^+e^- annihilation into hadrons and hadron production in τ decays [7].

The threshold region gives

$$a_\mu(\text{threshold}) \sim c_1 \left(\frac{\alpha}{\pi}\right)^2 \frac{m_\mu^2}{m_\pi^2} \tag{14}$$

i.e. a parametrical enhancement in the chiral limit. Numerically, however, this is not a leading contribution,

$$a_\mu^{\rm had,LO}(4m_\pi^2 \leq s \leq m_\rho^2/2) \approx 400\times 10^{-11} \tag{15}$$

Compare with the ρ peak,

$$a_\mu^{\rm had,LO}(\rho) = \frac{m_\mu^2\,\Gamma(\rho\to e^+e^-)}{\pi\, m_\rho^3} \approx 5000\times 10^{-11} \tag{16}$$

This contribution is enhanced by N_c,

$$a_\mu(\rho) \sim c_2 \left(\frac{\alpha}{\pi}\right)^2 N_c \, \frac{m_\mu^2}{m_\rho^2} \tag{17}$$

What is a lesson from this exercise? We see that the large N_c enhancement prevails over chiral one.

3 Light-by-light

The $\gamma^*\gamma^* \to \gamma^*\gamma$ amplitude is not accessible experimentally, a challenge for theorists. Parametrically the LbL contribution to a_μ can be presented in the form

$$a_\mu^{\rm LbL} \sim \left(\frac{\alpha}{\pi}\right)^3 \left[c_1 \, \frac{m_\mu^2}{m_\pi^2} + c_2 N_c \, \frac{m_\mu^2}{m_\rho^2}\right] \tag{18}$$

similar to $a_\mu^{\rm had,LO}$ above. The first, chirally enhanced term, is due to the loops of charged pion presented in Fig. 5a, the second, N_c-enhanced, term is due to exchanges of π^0 and heavier resonances, Fig. 5b. The chirally enhanced

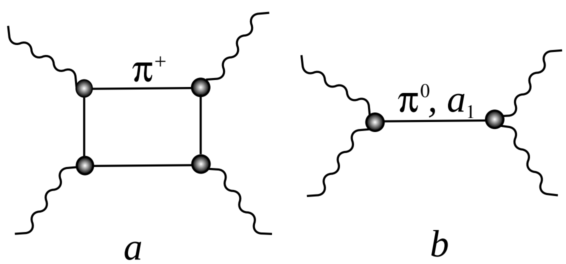

Figure 5: Hadronic contributions to the light-by-light scattering: (a) charged pion loop, (b) exchange of neutral pion and other resonances.

Figure 6: The π^0 pole part of light-by-light contribution to $g-2$

contribution does not result in large number, it is actually rather small [8],

$$a_\mu^{\rm LbL}(\text{pion box}) \approx -4\times 10^{-11} \tag{19}$$

(a larger number -19×10^{-11} was obtained in [9]) similarly to the hadronic polarization case above. The π^0 pole part of LbL, see Fig. 6, contains besides

N_c the chiral enhancement in the logarithmic form [10],

$$a_\mu^{\rm LbL}(\pi^0) = \left(\frac{\alpha}{\pi}\right)^3 N_c\, \frac{m_\mu^2\, N_c}{48\pi^2 F_\pi^2}\, \ln^2 \frac{m_\rho}{m_\pi} + \dots\ . \tag{20}$$

The π^0 pole part was studied in Refs. [8, 9, 11]. It was shown there that the logarithmically enhanced term is not sufficient, the subleading terms are equally important. Numerically [10]

$$a_\mu^{\rm LbL}(\pi^0) = 56\times 10^{-11}\,, \qquad\qquad \text{Knecht, Nyffeler.} \tag{21}$$

Using constraints from the Operator Product Expansion (OPE) we will show that it is underestimated value. The difference can be formulated as an absence of form factor suppression in $\gamma\gamma^*\pi^0$ vertex containing the soft photon (external field).

4 OPE constraints

The photons in the LbL amplitude are described by $\epsilon_i^\mu(q_i)$, $i = 1, 2, 3, 4$, $\sum q_i = 0$ where ϵ_4 represents the external magnetic field $f^{\gamma\delta} = q_4^\gamma \epsilon_4^\delta - q_4^\delta \epsilon_4^\gamma$, $q_4 \to 0$. The LbL amplitude is

$$\begin{aligned}\mathcal{M} \;&=\; \alpha^2 N_c\, {\rm Tr}\,[\hat{Q}^4]\, \mathcal{A} = \alpha^2 N_c\, {\rm Tr}\,[\hat{Q}^4]\, \mathcal{A}_{\mu_1\mu_2\mu_3\gamma\delta}\epsilon_1^{\mu_1}\epsilon_2^{\mu_2}\epsilon_3^{\mu_3} f^{\gamma\delta} \\ &=\; -e^3 \int {\rm d}^4x\, {\rm d}^4y\, {\rm e}^{-iq_1x - iq_2y}\, \epsilon_1^{\mu_1}\epsilon_2^{\mu_2}\epsilon_3^{\mu_3} \langle 0| T\,\{ j_{\mu_1}(x)\, j_{\mu_2}(y)\, j_{\mu_3}(0)\}\,|\gamma\rangle\,,\end{aligned} \tag{22}$$

where j_μ is the electromagnetic current, $j_\mu = \bar{q}\,\hat{Q}\gamma_\mu q$, $q = \{u, d, s\}$. The amplitude depends on three Lorentz invariants: q_1^2, q_2^2, q_3^2.

Consider the Euclidian range $q_1^2 \approx q_2^2 \gg q_3^2$. We can use OPE for the currents that carry large momenta q_1, q_2,

$$\begin{aligned}&i\int {\rm d}^4x\, {\rm d}^4y\, {\rm e}^{-iq_1x - iq_2y}\, T\,\{ j_{\mu_1}(x), j_{\mu_2}(y)\} = \\ &\int {\rm d}^4z\, {\rm e}^{-i(q_1+q_2)z}\, \frac{2i}{\hat{q}^2}\, \epsilon_{\mu_1\mu_2\delta\rho}\, \hat{q}^\delta j_5^\rho(z) + \cdots\ .\end{aligned} \tag{23}$$

Here $\hat{q} = (q_1 - q_2)/2$ and the axial current $j_5^\rho = \bar{q}\,\hat{Q}^2\gamma^\rho\gamma_5\, q$ is the linear combination of the isovector $j_{5\rho}^{(3)} = \bar{q}\,\lambda_3\gamma^\rho\gamma_5\, q$, the hypercharge $j_{5\rho}^{(8)} = \bar{q}\,\lambda_8\gamma^\rho\gamma_5\, q$

and the singlet $j_{5\rho}^{(0)} = \bar{q}\,\gamma^\rho\gamma_5\,q$ currents,

$$j_{5\rho} = \sum_{a=3,8,0} \frac{\mathrm{Tr}\,[\lambda_a \hat{Q}^2]}{\mathrm{Tr}\,[\lambda_a^2]}\, j_{5\rho}^{(a)} \tag{24}$$

This OPE is illustrated in Fig. 7,

Figure 7: The operator product expansion of two hard vector currents

The triangle amplitude

$$T^{(a)}_{\mu_3\rho} = i\,\langle 0|\int \mathrm{d}^4 z\, \mathrm{e}^{iq_3 z} T\{j^{(a)}_{5\rho}(z)\, j_{\mu_3}(0)\}|\gamma\rangle \tag{25}$$

is expressed kinematically via two scalar amplitudes

$$T^{(a)}_{\mu_3\rho} = -\frac{ie\,N_c \mathrm{Tr}\,[\lambda_a \hat{Q}^2]}{4\pi^2}\Big\{ w_L^{(a)}(q_3^2)\, q_{3\rho} q_3^\sigma \tilde{f}_{\sigma\mu_3} + $$
$$+ w_T^{(a)}(q_3^2)\Big(-q_3^2 \tilde{f}_{\mu_3\rho} + q_{3\mu_3} q_3^\sigma \tilde{f}_{\sigma\rho} - q_{3\rho} q_3^\sigma \tilde{f}_{\sigma\mu_3}\Big)\Big\}\,. \tag{26}$$

The longitudinal function w_L is associated with the pseudoscalar mesons exchange, while the transversal function w_T represents the pseudovector mesons exchange.

In perturbation theory for massless quarks

$$w_L^{(a)}(q^2) = 2 w_T^{(a)}(q^2) = -\frac{2}{q^2}\,. \tag{27}$$

Nonvanishing w_L is the signature of the axial Adler–Bell–Jackiw anomaly [12]. Moreover, for nonsinglet $w_L^{(3,8)}$ it is the *exact* QCD result, no perturbative (Adler–Bardeen theorem [13]) as well as nonperturbative ('t Hooft consistency condition [14]) corrections. So the pole behavior is preserved all way down to small q^2 where the pole is associated with Goldstone mesons π^0, η. Comparing the pole residue we get the famous ABJ result

$$g_{\pi\gamma\gamma} = \frac{N_c \mathrm{Tr}\,[\lambda_3 \hat{Q}^2]}{}.16\pi^2\, F_\pi \tag{28}$$

There exists [2] the nonrenormalization theorem for w_T as well but only in respect to perturbative corrections. Higher term in the OPE does not vanish in this case, they are responsible for shift of the pole $1/q^2 \to 1/(q^2-m^2)$.

Combining we get

$$\mathcal{A}_{\mu_1\mu_2\mu_3\gamma\delta} f^{\gamma\delta} = \frac{8}{\hat{q}^2}\epsilon_{\mu_1\mu_2\delta\rho}\hat{q}^\delta \sum_{a=3,8,0} W^{(a)} \Big\{ w_L^{(a)}(q_3^2)\, q_3^\rho q_3^\sigma \tilde{f}_{\sigma\mu_3} + w_T^{(a)}(q_3^2)\Big(-q_3^2 \tilde{f}^\rho_{\mu_3} + q_{3\mu_3} q_3^\sigma \tilde{f}^\rho_\sigma - q_3^\rho q_3^\sigma \tilde{f}_{\sigma\mu_3}\Big)\Big\} + \cdots , \qquad (29)$$

where the weights $W^{(3)} = 1/4,\ W^{(8)} = 1/12,\ W^{(0)} = 2/3$.

5 The model

Now we can formulate the model for the LbL amplitude which interpolates between pseudoscalar and pseudovector poles and the correct asymptotic behavior at large momenta,

$$\begin{aligned}
&\mathcal{A} = \mathcal{A}_{\rm PS} + \mathcal{A}_{\rm PV} + \text{permutations},\\
&\mathcal{A}_{\rm PS} = \sum_{a=3,8,0} W^{(a)} \phi_L^{(a)}(q_1^2,q_2^2)\, w_L^{(a)}(q_3^2)\, \{f_2\tilde{f}_1\}\{\tilde{f}f_3\},\\
&\mathcal{A}_{\rm PV} = \sum_{a=3,8,0} W^{(a)} \phi_T^{(a)}(q_1^2,q_2^2)\, w_T^{(a)}(q_3^2)\, \Big(\{q_2 f_2\tilde{f}_1\tilde{f}f_3q_3\}\\
&\qquad + \{q_1 f_1\tilde{f}_2\tilde{f}f_3q_3\} + \frac{q_1^2+q_2^2}{4}\{f_2\tilde{f}_1\}\{\tilde{f}f_3\}\Big).
\end{aligned} \qquad (30)$$

For π^0

$$\begin{aligned}
&w_L^{(3)}(q^2) = \frac{2}{q^2+m_\pi^2},\\
&\phi_L^3(q_1^2,q_2^2) = \frac{N_c}{4\pi^2F_\pi^2}\, F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2)\\
&= \frac{q_1^2q_2^2(q_1^2+q_2^2) - h_2 q_1^2q_2^2 + h_5(q_1^2+q_2^2) + (N_cM_1^4M_2^4/4\pi^2F_\pi^2)}{(q_1^2+M_1^2)(q_1^2+M_2^2)(q_2^2+M_1^2)(q_2^2+M_2^2)}.
\end{aligned} \qquad (31)$$

Following the form factor analysis by Knecht and Nyffeler [10] $M_1 = 769$ MeV, $F = 6.93\ \mathrm{GeV}^4$. They did not fix h_2 and put $h_2 = 0$ for the central value. Actually, it is fixed by the old QCD sum rule analysis [15], $h_2 \approx -10\ \mathrm{GeV}^2$.

It results in

$$a_\mu^{\pi^0} = 76.5 \times 10^{-11}\,, \qquad a_\mu^{\rm PS} = 114(10) \times 10^{-11}\,. \tag{32}$$

A similar analysis for pseudovector exchange gives

$$a_\mu^{\rm PV} = 20.5(5) \times 10^{-11}\,, \tag{33}$$

and finally

$$a_\mu^{\rm LbL} = 134(25) \times 10^{-11}\,. \tag{34}$$

6 Summary

The hadronic light-by-light scattering contribution to a_μ is shown to be larger than previous estimates. We cannot claim any significant reduction in the theoretical uncertainty although believe that the shift $\approx 50 \times 10^{-11}$ in the central value is real,

$$a_\mu^{\rm LbL} = 134(25) \times 10^{-11}\,. \tag{35}$$

In terms of comparison with the experimental value it means that

$$a_\mu^{\rm exp} - a_\mu^{\rm th} = \begin{cases} (220 \pm 100) \times 10^{-11} & (2.2\,\sigma), \quad e^+e^- \text{ based}, \\ (76 \pm 100) \times 10^{-11} & (0.8\,\sigma), \quad \tau \text{ based}. \end{cases} \tag{36}$$

Update: The recent data from KLOE at DAΦNE [16] (they use the radiative return to get a lower energy) are consistent the CMD-2 data in Novosibirsk. Although the discrepancy with τ data remains unexplained the authors of Ref. [7] made the new analysis putting aside the τ data but including the KLEO data,

$$a_\mu^{\rm exp} - a_\mu^{\rm th} = (236 \pm 92) \times 10^{-11} \qquad (2.6\,\sigma). \tag{37}$$

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