Any order generalization of BLM procedure in QCD

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Abstract

The BLM procedure is sequentially extended for any fixed order of the perturbation QCD and the obtained reformed perturbation series is equivalent to a continued fraction. A generalization of this procedure is developed which provides one with a certain mechanism of the FAC prescription. This generalized BLM procedure is applied to Adler function D in N^{3}LO and partially in N^{4}LO. The final effects of 2(3)–loop BLM improvement for D and $R_{e^+e^-}$ functions are discussed.

1 Introduction

The goal of the article is to extend the well-known Brodsky, Lepage and Mackenzie (BLM) procedure [1] of scale setting for any order of pQCD. Let me start with an appropriate citation: "...One, therefore, has to address the question of what is the "best" choice for μ^2 within a given scheme, usually $\overline{\text{MS}}$. There is no definite answer to this question – higher-order corrections do not "fix" the scale, rather they render the theoretical predictions less sensitive to its variation" (I. Hinchliffe, Particle Data Group booklet [2]). It

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will be shown that higher orders of pQCD in the $\overline{\mathrm{MS}}$–scheme unambiguously determine the new scales in the BLM prescription sense in contrast to the premonitory citation. Certainly, the effects of the coupling renormalization encoded in β-function coefficients are absorbed into the set of the eigen-scales μ_i^2 of the couplings $a_i = a_s(\mu_i^2)$ in any order of pQCD. To simplify the analysis of the structure of radiation corrections the renormalization-scheme invariant quantities like Adler function (D) and so on, see [6] for review, have been considered here. The procedure of the absorbtion is constructed in Sec.2, 3 and includes as a partial case the "bubble approximation" elaborated in [3, 4, 5]. The corresponding new perturbation series is obtained in Sec.3, so that the initial BLM suggestion [1] appears to be completed by the sequential BLM (sBLM) procedure. The construction of this sBLM is based on the visible relation between the QCD β–function coefficients b_i, $b_i \sim b_0^{i+1}$; the hierarchy of the contributions of coupling renormalization to the perturbation coefficients at every order of a_s is based on this power law. This detailed hierarchy requires a matrix representation for the perturbation expansion (PE) rather than the standard series. The mentioned power relation works at least up to the last known coefficient b_3 (at $N_f = 0 \div 5$), instead of the usually discussed proposition of the so-called "large–b_0" limit (at $-N_f \gg 1$, in the case, *e.g.* , $b_0 \sim b_1$). Of course, the power law should be destroyed somewhere in the higher orders of the PE when its expected factorial explosion starts.

Let me stress that the procedure can be formulated in terms of the dynamic characteristics only, the β–function coefficients, rather than in terms of certain $\mathrm{SU}(3)_c$–Casimirs that may appear at an intermediate stage. Therefore, I try to avoid using N_f powers in the consideration everywhere it is possible. This β–function expansion is performed for the 4-loop D-function (Appendix A), then the sBLM procedure is applied to this D-function in Sec.4 to highlight the advantages and disadvantages of the procedure in the case of this physical quantity.

The sBLM procedure may not be related to the improvement of the perturbation series. The next goal here is to supply the sBLM with the mechanism *a'la* Fast Apparent Convergence (FAC) [10] to improve the convergence of the series. This machinery, using the eigen-scales of the sBLM, is considered in Sec.4-5.

2 BLM task, the first stage of generalization

Preliminary. Let us consider the formal perturbation series $s(a)$ for the two-point amplitudes at the external momentum Q^2. The coupling $a \equiv a_s(\mu^2) = \alpha_s(\mu^2)/(4\pi)$ is normalized at the same external scale $\mu^2 = Q^2$. In the case the coefficients of the expansion d_n are the numbers in the *MS-like* schemes, due to the cancel of the logarithms $\ln(Q^2/\mu^2)$ there. However, the constant parts ($\ln(C)$) of these log's accompanied by the β-function coefficients are left in d_n and we shall manage just with these traces. For further convenience we introduce a new scaled expansion parameter $A = |b_0|a$,

$$s(a) = d_0 + \sum_{n=1} d_n a^n \equiv S(A) = d_0 + \frac{d_1}{b_0} \cdot \sum_{n=1} D_n A^n \qquad (1)$$
$$\text{with} \quad D_1 = 1,$$

and new coefficients $D_i = \dfrac{d_i}{d_1 b_0^{i-1}}$ that simplify intermediate calculations and will help us maintain contact with the "large b_0" limit, $b_0 \gg 1, \quad A \lesssim 1$. Note that in the real world, below the c-quark threshold (at $N_f = 3$) we have $b_0 = 9 \gg 1$ and $A(\mu^2) \equiv \alpha(\mu^2)\dfrac{b_0}{4\pi} \approx 0.32 < 1$ for the NLO level at $\mu^2 = 1\mathrm{GeV}^2$. The running of the coupling $A \to \bar{A}(t)$ (or $a \to \bar{a}(t)$) follows the renormgroup (RG) equation

$$\frac{d}{dt}\bar{A} \equiv B(\bar{A}) = -\left(\bar{A}^2 + c_1\bar{A}^3 + c_2\bar{A}^4 + \ldots\right); \; c_i = \frac{b_i}{b_0^{i+1}}; \qquad (2)$$

where $B(A)$-the modified β-function and $t = \ln\left(\frac{Q^2}{\Lambda^2}\right)$ is a natural variable for MS-like schemes.

The β-function structure of the perturbation coefficients. The standard BLM is based on the evident structure of NLO coefficient $d_2 = b_0 d_2[1] + d_2[0]$; the first term appears due to one-loop a-renormalization. In N^{2}LO the a-renormalization appearing from one gluon line generates contributions proportional to $a^3 b_0^2$, $a^3 b_1$, and from the next (this) gluon line generates a contribution $\sim a^3 b_0$ at the background of the first (next) one, correspondingly. The final representation for d_3 looks like an expansion in power series in $b_0, b_1, \ldots$

$$d_3 = d_1\left(b_0^2 d_3[2,0] + b_1 d_3[0,1] + b_0 d_3[1,0] + d_3[0]\right), \qquad (3)$$

where the first argument n_0 of the expansion coefficients $d_3[n_0, n_1, \ldots]$ corresponds to the power of b_0, and the second one n_1 – to the power of b_1, *etc.* The coefficient $d_n[0]$ corresponds to the so–called [18] "genuine" corrections with $n_i = 0$ for the all possible b_i powers. Moreover, if all the arguments of the coefficient $d_n[\ldots, m, 0, \ldots, 0]$ on the right of m are 0, then we shall omit these arguments for simplicity and write $d_n[\ldots, m]$ hereinafter. In N^{3}LO order the a renormalization generates contributions $\sim a^4 b_0^3,\ a^4 b_0 b_1,\ a^4 b_2$ and the contributions $\sim a^4 b_0^2,\ a^4 b_1,\ a^4 b_0$ from the mixing of renormalization of a from the different sources and $\sim a^4$ for the "genuine" corrections. The d_4 coefficient looks in this notation like

$$
\begin{aligned}
d_4 \;=\; & d_1 \left(b_0^3 d_4[3] + b_1 b_0 d_4[1,1] + b_2 d_4[0,0,1] + \right. \\
& \left. \qquad b_0^2 d_4[2] + b_1 d_4[0,1] + b_0 d_4[1] + d_4[0]\right) .
\end{aligned} \tag{4}
$$

The same ordering of the β-function elements holds for all the next d_n. It is convenient for our purposes to present this "β-structure" for the "scaled" variables $\bar{A},\ D_i$; the D_i coefficients have an evident form (presented up to A^5 order)

$$
\begin{aligned}
&\bar{A}^1(t) \qquad D_1 = \mathbf{1}; \\
&\bar{A}^2(t) \qquad D_2 = \mathbf{d_2[1]} + \frac{1}{b_0} \cdot d_2[0]; \\
&\bar{A}^3(t) \qquad D_3 = \mathbf{d_3[2]} + \mathbf{c_1 d_3[0,1]} + \frac{1}{b_0} \cdot \left(d_3[1] + \frac{1}{b_0} d_3[0] \right); \\
&\bar{A}^4(t) \qquad D_4 = \mathbf{d_4[3]} + \mathbf{c_1 d_4[1,1]} + \mathbf{c_2 d_4[0,0,1]} + \\
&\qquad\qquad\qquad \frac{1}{b_0} \cdot \left(d_4[2] + c_1 d_4[0,1] + \frac{1}{b_0} \cdot \left(d_4[1] + \frac{1}{b_0} d_4[0] \right) \right); \\
&\bar{A}^5(t) \qquad D_5 = \mathbf{d_5[4]} + \mathbf{c_1 d_5[2,1]} + \mathbf{c_1^2 d_5[0,2]} + \mathbf{c_2 d_5[1,0,1]} \\
&\qquad\qquad\qquad + \mathbf{c_3 d_5[0,0,0,1]} \\
&\qquad\qquad\qquad + \frac{1}{b_0} \cdot \left(d_5[3] + c_1 d_5[1,1] + c_2 d_5[0,0,1] \right. \\
&\qquad\qquad\qquad \left. + \frac{1}{b_0} \cdot \left(d_5[2] + \ldots \frac{1}{b_0} \cdot (\ldots) \right) \right);
\end{aligned} \tag{5}
$$

where c_i are defined in Eq.(2). Here we do not discuss how to derive this representation for the known multi-loop results; we suggest that the elements of the structure in Eq.(5) have already been obtained. The first column of the coefficients $\mathbf{d_n}[\mathbf{n-1}]$ in Eq.(5) corresponds to the "bubble approximation" that includes the contributions from the diagrams with the maximum numbers of the bubbles. However, there are other unsuppressed contributions in every row of the "table" of Eq.(5). Really, the known $c_i \sim 1$ for QCD $\overline{\text{MS}}$-scheme (at $N_f = 3$),

$$c_1 \approx 0.79; \quad c_2 \approx 0.88; \quad c_3 \approx 1.9; \quad c_4 =?; \tag{6}$$

therefore, one has no reason to neglect the other terms, emphasized here in the bold type. To extend our final results far more broadly, we shall suggest the same estimates, $c_i \sim 1$, for the unknown required coefficients as well (see the discussion in Introduction).

We face two different expansion parameters here, $\bar{A}$ for the lines and b_0^{-1} for the horizontal direction. To simplify the handling of these parameters, it is convenient to invent the notation $\bar{A}^i \cdot y_{ij} \cdot b_0^{-j+1}$ for the corresponding contributions and $D_i = y_{ij} \cdot b_0^{-j+1}$ for their coefficients. The $Y = ||y_{ij}||$ is a triangular matrix with the diagonal "genuine" terms $y_{nn} \equiv d_n[\mathit{0}]$ that are maximum suppressed by the b_0^{-1} powers in D_n, while the unsuppressed terms (emphasized in bold type in Eq.(5)) are contained in the first column of the matrix Y

$$y_{11} \equiv \mathbf{1}; \tag{7}$$
$$y_{21} = \mathbf{d_2}[\mathbf{1}]; \tag{8}$$
$$y_{31} = \mathbf{d_3}[\mathbf{2}] + \mathbf{c_1 d_3}[\mathbf{0}, \mathbf{1}]; \tag{9}$$
$$y_{41} = \mathbf{d_4}[\mathbf{3}] + \mathbf{c_1 d_4}[\mathbf{1}, \mathbf{1}] + \mathbf{c_2 d_4}[\mathbf{0}, \mathbf{0}, \mathbf{1}]; \tag{10}$$
$$\ldots \quad .$$

The y_{n1} originate from the renormalization of a single coupling/gluon line in the skeleton diagram.

BLM scheme for n-loop. Let us find a new pair $(t_1, A(t_1))$ to vanish all the "bold type" contributions y_{i1} and to accumulate them in the new expansion parameter $A(t_1)$

$$\begin{aligned} &\bar{A}(t) \to \bar{A}(t_1) \equiv A_1; \\ &t - t_1 \equiv \Delta_1 = \Delta_{1,0} + A_1 \cdot \Delta_{1,1} + A_1^2 \cdot \Delta_{1,2} + \ldots . \end{aligned} \tag{11}$$

Here the shift Δ_1 of the t to the first intrinsic scale t_1 is found in the form of a perturbation series in A_1 (that has first been suggested in [15]). The corresponding procedure consist in the re-expansion of Exp.(5) in the new coupling A_1 and rearrangement of the power series. Following the RG law for the coupling $\bar{A}(t) = \bar{A}(\Delta_1, A_1)$ and expanding it in Δ_1 one obtains

$$\bar{A}(t) = \bar{A}(t - t_1, A_1) = A_1 - B(A_1)\Delta_1 + B'(A_1)B(A_1)\frac{\Delta_1^2}{2} + \dots \quad .$$

Substituting this expansion together with the expansion for Δ_1 in Eq.(11) into Exp.(5) one arrives at the rearranged series

$$\begin{aligned}
\bar{A}^1 D_1 \to &\quad \bar{A}_1^1 \cdot \quad 1; \\
\bar{A}^2 D_2 \to &\quad \bar{A}_1^2 \cdot \quad D_2 - 1\Delta_{1,0}; && (12)\\
\bar{A}^3 D_3 \to &\quad \bar{A}_1^3 \cdot \quad D_3 - 2\Delta_{1,0}\cdot D_2 - \Delta_{1,0}c_1 + \Delta_{1,0}^2 - \Delta_{1,1}; && (13)\\
\bar{A}^4 D_4 \to &\quad \bar{A}_1^4 \cdot \quad D_4 - 3\Delta_{1,0}\cdot D_3 + \left(3\Delta_{1,0}^2 - 2c_1\Delta_{1,0}\right) D_2 - \\
&\qquad\qquad c_2\Delta_{1,0} + \frac{5}{2}c_1\Delta_{1,0}^2 - \Delta_{1,0}^3 + \\
&\qquad\qquad (2\Delta_{1,0} - 2D_2 - c_1)\,\Delta_{1,1} - \Delta_{1,2}; && (14)\\
A^{n+1}D_{n+1} \to &\quad A_1^{n+1} \cdot \quad D_{n+1} - n\Delta_{1,0}\cdot D_n + \dots,
\end{aligned}$$

The generalized BLM requires that the y_{i1} *contributions should cancel* for every order $\bar{A}_1^i$ in the set of Eqs. (12, 13, 14, ...). This requirement completely determines the partial "scales" $\Delta_{1,i}$ from the set of equations; here we write few important coefficients for the discussion, the first of them, Eq.(15), corresponds to the standard BLM scale setting,

$$\begin{aligned}
\Delta_{1,0} &= y_{21} = d_2[1]; && (15)\\
\Delta_{1,1} &= y_{31} - (y_{21})^2 - c_1 y_{21} \\
&= d_3[2] - d_2^2[1] + c_1\left(d_3[0,1] - d_2[1]\right); && (16)\\
\Delta_{1,2} &= y_{41} - 3y_{31}y_{21} - 2(y_{21})^3 - c_1\cdot\dots \\
&= d_4[3] - 3d_2[1]d_3[2] + 2(d_2[1])^3 + c_1\cdot\dots && (17)
\end{aligned}$$

Note that NLO BLM correction in Eq.(16) cancels at the special conditions $d_3[2] = (d_2[1])^2$, $d_3[0,1] = d_2[1]$. First of them corresponds to the geometric progression for the leading *log* of the RG law, while the second one corresponds to sub-leading *log* cancellation. If one applies these conditions to $\Delta_{1,2}$ in Eq.(17), one obtains again the evident "geometric" condition, $d_4[3] = (d_2[1])^3$, for the cancel of the leading *log* part at the next step and so on.

In that way one can rearrange the first column y_{i1} into Δ_1 step by step for any fixed order of the PE. As the result of the procedure the initial series, Eq.(5), can be reduced to the new one that contains only one unsuppressed term $\bar{A}_1 \cdot 1$, the first diagonal term in Eq.(18), all the other terms are suppressed by the powers of b_0^{-1},

$$\begin{aligned}
\bar{A}^1 D_1 \to \quad & \bar{A}_1^1 \cdot \quad 1; \\
\bar{A}^2 D_2 \to \quad & \bar{A}_1^2 \cdot \quad 0 + \frac{y_{22}}{b_0}; \\
\bar{A}^3 D_3 \to \quad & \bar{A}_1^3 \cdot \quad 0 + \frac{1}{b_0}\left(y_{32} - 2y_{21}\ y_{22}\right) + \frac{y_{33}}{b_0^2}; \\
\bar{A}^4 D_4 \to \quad & \bar{A}_1^4 \cdot \quad 0 + \frac{1}{b_0}\big(y_{42} - 3y_{21}\ y_{32} + y_{22}[5y_{21}^2 - 2y_{31}]\big) + \\
& \qquad\qquad \frac{1}{b_0^2}\left(y_{43} - 3y_{21}\ y_{33}\right) + \frac{y_{44}}{b_0^3}; \\
A^n D_n \to \quad & A_1^n \cdot \quad 0 + \frac{1}{b_0}\left(y_{n2} - \ldots\right)\ldots \qquad .
\end{aligned} \tag{18}$$

At this stage the matrix Y transforms into the new matrix $Y^{(1)}$, the first column of which is now $y_{1i}^{(1)} = \delta_{1i}$ and the other few elements are presented in Eq.(18). The first BLM stage result can be rewritten in the form of the matrix representation, $\sum_{i>j} \bar{A}^i \cdot y_{ij} \cdot b_0^{-j+1} \equiv \bar{A}\mathcal{A}^{+} Y B$, where $\mathcal{A} = \left(1, \bar{A}, \bar{A}^2, \ldots\right), \mathcal{A}_i = \left(1, \bar{A}_i, \bar{A}_i^2, \ldots\right)$, $B = \left(1, b_0^{-1}, b_0^{-2}, \ldots\right)$,

$$\bar{A}\left(\mathcal{A}^{+} Y B\right)_n \stackrel{1\ \text{stage}}{\longrightarrow} \bar{A}_1\left(\mathcal{A}_1^{+} Y^{(1)} B\right)_n = \bar{A}_1\left(1 + \bar{A}_1 (\mathcal{A}_1^{+} Y^{(1)} B)_{n-1}\right).$$

The single unsuppressed (diagonal) term, 1, is extracted in the r.h.s. in the parentheses, while the second term there is formed by the power b_0^{-1}–suppressed minor of the matrix $Y^{(1)}$.

3 Sequential BLM procedure, next stages

Let us continue to put the matrix $Y^{(1)}$ into the diagonal form by reforming its *second column*. Now we deal with its $(Y^{(1)})_{(n-1)}$-minor contained only the b_0^{-1}–suppressed terms,

$$A_1 (\mathcal{A}_1^{+} Y^{(1)} B)_n = \bar{A}_1\left(1 + \bar{A}_1 (\mathcal{A}_1^{+} Y^{(1)} B)_{n-1}\right).$$

The minor elements are presented in the rectangular on the right-hand side of Table 1. The first power b_0^{-1}–suppressed terms of the minor are emphasized in the bold type there. Repeating the same procedure as on the first stage under the column $y_{i2}^{(1)}$ now, we rearrange again these terms into the new expansion parameter $A(t_2)$ at the new "scale" t_2 obtained from the t_1

$$\begin{aligned} &\bar{A}(t_1) \to \bar{A}(t_2) \equiv A_2; \\ &t_1 - t_2\ werearrangeagaina_{2,1} + A_2^2 \cdot \Delta_{2,2} + \dots, \end{aligned} \tag{19}$$

The first equalities in Eqs. (15-17) remain valid also for the partial scales $\Delta_{2,m}$ appearing from the $y_{i2}^{(1)}$ with the evident shifting of all the indices by 1. Therefore, using Eqs.(18) to determine $y_{i2}^{(1)}$ and taking the common factor

Table 1: The structure of $\bar{A}\cdot\ \ \bar{A}^{i-1} y_{ij}^{(1)} b_0^{1-j}$ contributions

$A^1 D_1 \to$	$\bar{A}_1\cdot$	1	1
$A^2 D_2 \to$	$\bar{A}_1\cdot$	$\bar{A}^1(t_1)$	$0 + \dfrac{d_2[0]}{b_0} \cdot \mathbf{1}$
$A^3 D_3 \to$	$\bar{A}_1\cdot$	$\bar{A}^2(t_1)$	$0 + \dfrac{d_2[0]}{b_0} \mathbf{\tilde{y}_{32}^{(1)}} + \dfrac{d_3[0]}{b_0^2}$
$A^4 D_4 \to$	$\bar{A}_1\cdot$	$\bar{A}^3(t_1)$	$0 + \dfrac{d_2[0]}{b_0} \mathbf{\tilde{y}_{42}^{(1)}} + \dfrac{1}{b_0^2} y_{43}^{(1)} + \dfrac{d_4[0]}{b_0^3}$
$A^5 D_5 \to$	$\bar{A}_1\cdot$	$\bar{A}^4(t_1)$	$0 + \dfrac{d_2[0]}{b_0} \mathbf{\tilde{y}_{52}^{(1)}} + \dfrac{1}{b_0^2} y_{53}^{(1)} + \dfrac{1}{b_0^3} y_{54}^{(1)} + \dfrac{d_5[0]}{b_0^4}$
	$\dots$	$\dots$	$\dots$

$\dfrac{d_2[0]}{b_0}$ to normalize the elements ($\mathbf{\tilde{y}_{n2}^{(1)}} = y_{n2}^{(1)}/y_{22}$) one can obtain for $\Delta_{2,m}$,

$$\Delta_{2,0} = \mathbf{\tilde{y}_{32}^{(1)}} = \frac{y_{32}}{y_{22}} - 2y_{21} = \frac{d_3[1]}{d_2[0]} - 2d_2[1]; \tag{20}$$

$$\Delta_{2,1} = \mathbf{\tilde{y}_{42}^{(1)}} - \left(\mathbf{\tilde{y}_{32}^{(1)}}\right)^2 - c_1 \mathbf{\tilde{y}_{32}^{(1)}} \tag{21}$$

$$\dots$$

So column by column the chain of transformations $Y \to Y^{(1)} \to Y^{(2)} \to \ldots \to Y^{(n-1)}$ leads to the diagonalization of Table 1. At every of these stages one will obtain new coupling $A(t_i)$. The final result of this successive BLM procedure reduces the initial perturbation series, Eq.(1), to the special one

$$S(A) = d_0 + \frac{\bar{A}(t_1)}{b_0} \cdot d_1 \left\{1 + \frac{\bar{A}(t_2)}{b_0} d_2[0] \left\{1 + \frac{\bar{A}(t_3)}{b_0} d_3[0] \left\{1 + \ldots\right\}\right\}\right\} \quad (22)$$

containing only the "genuine" coefficients $d_i[0]$ accompanied by the corresponding coupling at its eigen-scale t_i, $t_i = t_1 - \Delta_1 - \ldots - \Delta_i$. Moreover, the series in powers $\bar{a}^n(t)$ transforms to the series in products $\prod_{i=1}^{n} \bar{a}(t_i)$. Note that Eq.(22) can be easily presented in the form *a'la* the continued fraction

$$S(A) = d_0 + \cfrac{\bar{a}_1 d_1[0]}{1 - \cfrac{\bar{a}_2 d_2[0]}{1 + \bar{a}_2 d_2[0] - \cfrac{\bar{a}_3 d_3[0]}{1 + \bar{a}_3 d_3[0] - \cfrac{\bar{a}_4 d_4[0]}{\ldots,}}}} \quad (23)$$

where $\bar{a}_i = \bar{a}(t_i)$. Of course, the final results, Eq.(22) (or Eq.(23)), look rather formal because the sBLM procedure constructed above disregards the perturbative applicability constraints for both the pairs $(A(t_i),\ t_i)$ and the new expansion coefficients $d_i[0]$. We meet this result with the N^{3}LO calculation of the D–function in Section 4.

4 NLO BLM procedure for the D function

The initial well-known series for D [11] can be rewritten by means of the β-function coefficients

$$\begin{aligned}
D &= 3 \sum_f Q_f^2 \left\{d_0 + d_1 \left[a + d_2 a^2 + d_3 a^3 + d_4 a^4 + \ldots\right]\right\}, \\
&\qquad d_0 = 1;\ d_1 = 3C_F; \\
d_2 &= b_0 \cdot d_2[1] + d_2[0]; \\
d_3 &= b_0^2 \cdot d_3[2] + b_1 \cdot d_3[0,1] + b_0 \cdot d_3[1] + d_3[0]; \qquad (24)\\
d_4 &= b_0^3 \cdot d_4[3] + b_0 b_1 \cdot d_4[1,1] + b_2 \cdot d_4[0,0,1] + b_0^2 \cdot d_4[2] + \ldots
\end{aligned}$$

A separate problem is to put D into this form; it is solved in Appendix A basing on the results obtained in [12] and on the partial results for d_4 in [16]. Note that the expressions for the expansion elements in (24) remain valid for the inclusion of light gluinos that contribute to the β-function. The explicit expressions for $d_3[m,n]$ are presented in Appendix A; below they are written numerically

$$d_2 = b_0 \cdot 0.69 \ + \ \frac{1}{3}; \tag{25}$$

$$\begin{aligned} d_3 = b_0^2 \cdot 3.104 \quad - \quad b_1 \cdot 1.2 \quad + b_0 \cdot 55.70 \\ + \left(-573.96 - 19.83 \frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)}\right), \end{aligned} \tag{26}$$

We substitute the value $b_0(N_f = 3) = 9$, $b_1(N_f = 3) = 64$ in (26) for illustration

$$d_3 = \ 251.1 \ - \ 76.8 \ + \ 501.3 \ + \ (-573.96 - 0) \approx \mathbf{101.9}, \tag{27}$$

to compare the contributions from different sources. Further, we shall apply the sBLM procedure to D step by step to remove, respectively, b_0-contribution at N^{2}LO; b_0^2 and b_1-contributions at N^{3}LO and so on. The results of sBLM will be analyzed at every step.

At the first standard step the BLM scale setting transforms the coefficients $d_2,\ d_3$ (compare with expressions (25-26)) and the coupling following

$$d_2 \ \longrightarrow \ \tilde{d}_2 = b_0 \cdot 0 + \frac{1}{3} \tag{28}$$

$$\begin{aligned} d_3 \ \longrightarrow \ \tilde{d}_3 = b_0^2 \left(d_3[2] + d_3[0,1]c_1 - d_2^2[1] - d_2[1]c_1\right) \\ +b_0(d_3[1] - 2d_2[0]d_2[1]) + d_3[0] \end{aligned} \tag{29}$$

$$= b_0^2 \ (\quad 2.1555 \quad - \quad 1.0251 \quad) \tag{30}$$

$$+b_0(\ 55.70 - 0.46 \quad) + \ldots \approx \mathbf{14.7} \tag{31}$$

$$A(t) \ \longrightarrow \ A(t_1); \ t - t_1 = \Delta_{1,0} = d_2[1] \approx 0.69 \tag{32}$$

One makes sure that the value of $b_0^2 \ y_{31}$ approximately reduces twice at the first step (at the same condition as for Eq.(27)) while the $b_0 \ y_{32}$ value practically does not change, the amount of all the terms is reduced to 14.7 in comparison with the initial value $d_3 \approx 101.9$ in Eq.(27). This strong cancel effect as well as the other features of the BLM steps appear due to the large and negative value of the genuine term $d_3[0]$.

At the next step of the stage the modified $\tilde{y}_{31}$ term in Eq.(29) is transferred into Δ_1 following Eqs.(15-16), respectively,

$$
\begin{aligned}
\tilde{d}_3 \quad &\to \quad \tilde{\tilde{d}}_3 = b_0^2 \cdot 0 + b_1 \cdot 0 \\
&\qquad\qquad + b_0 \cdot (d_3[1] - 2d_2[0]d_2[1]) + d_3[0] \approx -\mathbf{77} \qquad (33) \\
A(t) \quad &\to \quad A(t_1); \\
&\quad t - t_1 = \Delta_1 = d_2[1] + \\
&\qquad\qquad + A(t_1) \cdot \big(d_3[2] + d_3[0,1]c_1 - d_2^2[1] - d_2[1]c_1\big) \qquad (34) \\
&\qquad\qquad\qquad \Delta_1 \approx 0.69 + A(t_1) \cdot 1.13 \qquad (35)
\end{aligned}
$$

The value of $\tilde{\tilde{d}}_3 \approx -77$ becomes noticeably larger in absolute value than at the first step in Eq.(31), while the first perturbation correction to Δ_1 in Eq.(35) looks rather moderate and admissible. Here in Eq.(35) one can put $t_1 \approx t - d_2[1]$ for the A argument rather than to solve Eq.(34) with respect to t_1. The contents of d_4 in Eq.(4) also transforms following to Eq.(17). Based on the results in [16], that lead to $d_4[3] \approx 2.18$, one can predict the modification of the"bubble part" $d_4[3]$ of this coefficient,

$$d_4[3] \approx 2.18 \to d_4[3] - d_3[2]d_2[1] - 2d_2[1](d_3[2] - d_2[1]^2) \approx -3.3$$

that is not also improved in itself. So one can conclude that though the next step of sBLM is admissible due to the moderate size of correction to Δ_1, it does not improve the convergence of the perturbation series.

At last the second stage of the sBLM procedure for D,
1. is ruled out t_2 from the pQCD domain because $t_1 - t_2 = \Delta_{2,0} = d_3[1]/d_2[0] - 2d_2[1] \approx 166$ (!);
2. does not lead to the decrease in the $\tilde{\tilde{d}}_3$ term $\tilde{\tilde{d}}_3 \to \tilde{\tilde{\tilde{d}}}_3 = d_3[0] \approx -574$ due to the large value of the genuine term; compare the contributions of the different terms in Eq.(27).

Therefore, we would not perform the second stage at all and try another way to optimize $\tilde{d}_3$ after the first step. It is tempting not to remove the contribution $y_{31} = d_3[2] + c_1 d_3[0,1]$ completely, as we did at the second step above, but rearrange its $\mathbf{x}$–part, $\mathbf{x}y_{31}$, into the coupling renormalization and keep the positive $(\mathbf{1} - \mathbf{x})$–part in the rest to compensate the large and negative $d_3[0]$ contribution. This trick leads to the $\mathbf{x}$-dependent BLM transformation

(xBLM)

$$\begin{aligned}
\tilde{d_3} \quad &\rightarrow \quad \tilde{\tilde{d_3}} = b_0^2 \cdot (\mathbf{1} - \mathbf{x})(d_3[2] + c_1 d_3[0,1]) \\
&\qquad\quad + b_0 \cdot (d_3[1] - 2d_2[0]d_2[1]) + d_3[0], \\
t - t_1 \quad &= \quad \Delta_1 = d_2[1] + A(t_1) \cdot (\mathbf{x}(d_3[2] + d_3[0,1]c_1) \\
&\qquad\quad - d_2^2[1] - d_2[1]c_1) \, .
\end{aligned} \tag{36}$$

Let us set an "optimization" condition, say FAC, $\tilde{\tilde{d_3}} = 0$, to fix a certain value of $\mathbf{x}$. One makes sure that the perturbation corrections are improved for both $\tilde{\tilde{d_3}}$ and $\tilde{\Delta}_1$ (see the 5-6th columns in Table 2) in comparison with ones in Eq.(33, 35). The final result for D is reduced to

$$D = 3 \sum_f Q_f^2 \left\{ 1 + 3C_F \left[a(\tilde{t}_1) + \frac{1}{3} a^2(\tilde{t}_1) + 0 \right] \right\} , \tag{37}$$

where $t - \tilde{t}_1 = \tilde{\Delta}_1$ are presented in Table 2.

Table 2: the $\tilde{\tilde{d}}_3 = 0$ and $\tilde{\tilde{r}}_3 = 0$ conditions

N_f	$b_0(N_f)$		x	$t - \tilde{t}_1 = \tilde{\Delta}_1$	$\tilde{\Delta}_1(Q^2)$		x	$s - \tilde{s}_1 = \tilde{\Delta}_1$
3	9		0.56	$d_2[1] + a(\tilde{t}_1)b_0 \cdot 0.18$			1.84	$d_2[1] - a(\tilde{s}_1)b_0 \cdot 3.1$
4	$\frac{25}{3}$	$\tilde{\tilde{d}}_3 = 0$	0.24	$d_2[1] - a(\tilde{t}_1)b_0 \cdot 0.45$	$(Q^2 = 3\mathrm{GeV}^2)$ 0.58	$\tilde{\tilde{r}}_3 = 0$	2.56	$d_2[1] - a(\tilde{s}_1)b_0 \cdot 3.7$
5	$\frac{23}{3}$		-0.11	$d_2[1] - a(\tilde{t}_1)b_0 \cdot 1.19$	$(Q^2 = 26\mathrm{GeV}^2)$ 0.52		3.63	$d_2[1] - a(\tilde{s}_1)b_0 \cdot 4.48$

It is instructive to apply a similar procedure also to the observable quantity $R(s) = \sigma(e^+e^- \to h)/\sigma(e^+e^- \to \mu^+\mu^-)$ associated with the D–function

$$R(s) = D(s) - d_1\frac{\pi^2}{3}\cdot b_0^2\bar{a}^3 = 3\sum_f Q_f^2\left\{1 + 3C_F\left[\bar{a} + r_2\bar{a}^2 + r_3\bar{a}^3\right]\right\},$$

where $r_1 = d_1,\ r_2 = d_2,\ r_3 = d_3 - \pi^2/3\cdot b_0^2$ see, *e.g.* , [11]. Large and negative π^2-term arising due to the analytic continuation makes r_3 also negative (compare with Eq.(26)). As a result of the xBLM procedure $r_3 \to \tilde{r}_3$, the x-dependent term in $\tilde{r}_3$ transforms to $b_0^2\cdot(1-x)(d_3[2] + c_1 d_3[0,1] - \pi^2/3)$.

5 Generalized BLM procedure to improve the perturbation series

To generalize the sBLM procedure in the way mentioned in Sec. 4, let us introduce a lower triangular matrix $X = ||x_{ij}||,\ x_{ii} \equiv 1$ associated with the matrix Y. The element x_{ij} is the part of the contribution y_{ij}, $\ y_{ij}\ x_{ij}$, that should be involved into the coupling renormalization, while the remainder of the contribution is $y_{ij}\ \bar{x}_{ij}$, where $\bar{x}_{ij} \equiv 1 - x_{ij}$. The choice $X = \mathbf{1}$ returns us to the initial series, while the choice $\mathbf{x_{i1}} = 1$ corresponds to the first stage of the sBLM; compare second column in Table 3 with the Eq.(18). These additional free parameters x_{ij}, altogether $n(n-1)/2$ parameters in N^{n}LO of PE, allow one to perform a "fine turning" of the coefficients of the initial series. A more complicated structure of the final PE series is the price one should pay for such an improvement of the convergence of the series. In the case the "first stage coupling" has the form

$$\begin{aligned}
\bar{A}(t) \to \bar{A}(t_1) \equiv A_1;\ t - t_1 \equiv \Delta_1 &= \Delta_{1,0}(X) + A_1\cdot\Delta_{1,1}(X) + \ldots, \\
\Delta_{1,0}(X) &= y_{21}\mathbf{x_{21}} \qquad (38)\\
\Delta_{1,1}(X) &= y_{31}\mathbf{x_{31}} - 2(y_{21}\mathbf{x_{21}})y_{21} + \\
&\quad +(y_{21}\mathbf{x_{21}})^2 - (y_{21}\mathbf{x_{21}})c_1 \qquad (39)
\end{aligned}$$

At the second stage of the sBLM generalization the right corner of Table 3 is reformed with the coupling A_2, see Table 4

$$\begin{aligned}
&\bar{A}(t_1) \to \bar{A}(t_2) \equiv A_2; \qquad (40)\\
&t_1 - t_2 \equiv \Delta_2 = \Delta_{2,0}(X) + A_1\cdot\Delta_{2,1}(X) + \ldots,\\
&\Delta_{2,0}(X) = \frac{y_{32}}{y_{22}}\mathbf{x_{32}} - 2y_{21}\ x_{21}
\end{aligned}$$

Table 3: the first stage of the xBLM procedure

$\bar{A}^1 D_1 \to \bar{A}_1\cdot$	1	$+\,0$
$\bar{A}^2 D_2 \to \bar{A}_1\cdot$	$\bar{A}_1^1 y_{21}\bar{\mathbf{x}}_{\mathbf{21}}$	$+\,\bar{A}_1^1\dfrac{y_{22}}{b_0}$
$\bar{A}^3 D_3 \to \bar{A}_1\cdot$	$\bar{A}_1^2 y_{31}\bar{\mathbf{x}}_{\mathbf{31}}$	$+\,\bar{A}_1^2\dfrac{y_{22}}{b_0}\left(\dfrac{y_{32}}{y_{22}} - 2y_{21}\ \mathbf{x}_{\mathbf{21}}\right) + \bar{A}_1^2\dfrac{y_{33}}{b_0^2}$
$\bar{A}^n D_n \to \bar{A}_1\cdot$	$\bar{A}_1^{n-1} y_{n1}\bar{\mathbf{x}}_{\mathbf{n1}}$	$+\ldots$

Table 4: the second stage of the xBLM procedure

$\bar{A}^1 D_1 \to \bar{A}_1\cdot$	1	$+0$
$\bar{A}^2 D_2 \to \bar{A}_1\cdot$	$\bar{A}_1^1 y_{21}\bar{x}_{21}$	$+\,\bar{A}_2^1\dfrac{y_{22}}{b_0}$
$\bar{A}^3 D_3 \to \bar{A}_1\cdot$	$\bar{A}_1^2 y_{31}\bar{x}_{31}$	$+\,\bar{A}_2^2\dfrac{y_{22}}{b_0}\left(\dfrac{y_{32}\ \bar{\mathbf{x}}_{\mathbf{32}}}{y_{22}}\right) + \bar{A}_2^2\dfrac{y_{33}}{b_0^2}$
$\bar{A}^n D_n \to \bar{A}_1\cdot$	$\bar{A}_1^{n-1} y_{n1}\bar{x}_{n1}$	$+\ldots$

(i) The FAC setting at A^2 corresponds to the condition $(\bar{A}_1)^2(y_{21}\bar{x}_{21} + \dfrac{y_{22}}{b_0}) = 0$; (ii) If one restricts himself , say, the NLO BLM at A^3 (see Table 4) then one has 3 parameters, x_{21}, x_{31}, x_{32} to optimize the contributions A^2D_2 and A^3D_3, respectively,

$$A^2 D_2 \to C_2 \qquad = \bar{A}_1\left[\bar{A}_1 y_{21}\bar{x}_{21} + \bar{A}_2\frac{y_{22}}{b_0}\right]; \tag{41}$$

$$A^3 D_3 \to C_3 \qquad = \bar{A}_1\left[\bar{A}_1^2 y_{31}\bar{x}_{31} + \bar{A}_2^2\frac{y_{22}}{b_0}\left(\frac{y_{32}\ \bar{x}_{32}}{y_{22}}\right) + \bar{A}_2^2\frac{y_{33}}{b_0^2}\right]; \tag{42}$$

$$\text{where} \qquad \bar{A}_1 = \bar{A}(t-\Delta_1),\ \ \bar{A}_2 = \bar{A}(t-\Delta_1-\Delta_2). \tag{43}$$

The case discussed in Sec.4 in Table 2 corresponds to the partial solution of the above equations at $C_3 = 0$, $A_1 = A_2$ with $x_{21} = 1$, $x_{31} = \mathbf{x}$, $x_{32} = 0$. The set of solutions to Eqs.(41-43) at $C_2 = C_3 = 0$ (the FAC condition) with respect to x_{ij} can be obtained and analyzed numerically.

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A Representation for the D-function

The required β-function coefficients with the MSSM light gluinos [12, 19],

$$b_0\left(N_f, N_g\right) = \frac{11}{3}C_A - \frac{4}{3}\left(T_R N_f + \frac{N_g C_A}{2}\right); \tag{A.1}$$

$$b_1\left(N_f, N_g\right) = \frac{34}{3}C_A^2 - \frac{20}{3}C_A\left(T_R N_f + \frac{N_g C_A}{2}\right) -4\left(T_R N_f C_{\mathrm{F}} + \frac{N_g C_A}{2}C_A\right). \tag{A.2}$$

Adler function is known [12] for the MSSM light gluinos (N_g), $D(a, N_f, N_g)$. On the other hand, one can obtain the explicit functions $N_f = N_f(b_0, b_1)$ and $N_g = N_g(b_0, b_1)$ solving the set of equations (A.1,A.2) with respect to the variables N_f, N_g. Substituting this solution into $D(a, N_f, N_g)$, one arrives at the expansions, Eq.(A.3-A.7),

$$D(A) = 3\sum_f Q_f^2\left\{d_0 + \frac{d_1}{b_0}\cdot\left(D_1 A + D_2 A^2 + D_3 A^3 + \ldots\right)\right\},$$

$$D_1 = 1;\ D_2 = d_2[1] + \frac{1}{b_0}d_2[0];$$

$$D_3 = d_3[2] + c_1\cdot d_3[0,1] + \frac{1}{b_0}d_3[1] + \frac{1}{b_0^2}d_3[0] \tag{A.3}$$

The N_f^2–terms of d_4 have recently been calculated in [16], but this could not been used in our approach. It is impossible to separate the terms $b_2 d_4[0,0,1]$ and $b_0 b_1 d_4[1,1]$ that are of an order of $O(b_0^3)$ from the b_0^2–term, $b_0^2 d_4[2]$ that also contributes to the "N_f^2 projection".

$$\begin{aligned}
d_2[1] &= \frac{11}{2} - 4\zeta_3 \approx 0.691772;\\
d_2[0] &= \frac{C_A}{3} - \frac{C_F}{2} = \frac{1}{3}; && \text{(A.4)}\\
d_3[2] &= \frac{302}{9} - \frac{76}{3}\zeta_3 \approx 3.10345;\\
d_3[0,1] &= \frac{101}{12} - 8\zeta_3 \approx -1.19979; && \text{(A.5)}\\
d_3[1] &= C_A\left(\frac{3}{4} + \frac{80}{3}\zeta_3 - \frac{40}{3}\zeta_5\right) - \\
&\quad -C_F\left(18 + 52\zeta_3 - 80\zeta_5\right) \approx 55.7005 && \text{(A.6)}\\
d_3[0] &= \frac{1}{36}(523C_A^2 + 852C_AC_F - 414C_F^2) - 72C_A^2\zeta_3 + \\
&\quad + \frac{5}{24}(\frac{176}{3} - 128\zeta_3)\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)}\\
&\approx -573.9607 - 19.8326\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)} && \text{(A.7)}
\end{aligned}$$

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