Spectral Boundary Conditions in the Bag Model

A. A. Abrikosov, jr.*
117 259, Moscow, ITEP.

Abstract

We propose a reduced form of spectral boundary conditions for holding fermions in the bag in a chiral invariant way. Our boundary conditions do not depend on time and allow Hamiltonian treatment of the system. They are suited for studies of chiral phenomena both in Minkowski and Euclidean spaces.

Introduction

The two principal problems of QCD are confinement and spontaneous breaking of chiral invariance. Both phenomena take place in the strongly interacting domain where the theory becomes nonperturbative. Probably they are interrelated. However, usually they were considered separately. Up to now the spontaneous chiral invariance breaking (SCIB) was discussed mostly in the infinite space. It would be interesting to address the features of SCIB that appear due to localization of quarks in finite volume. In order to do that one needs to hold quarks in a chiral invariant way.

There exists an entire family of bag models. The famous MIT bag [1] successfully reproduced the spectrum and other features of hadrons. A generalization of the MIT model are so-called chiral bags [2, 3]. An apparent drawback of all these models is that the boundary conditions are explicitly chiral noninvariant.

*E-mail: PERSIK@ITEP.RU

Attempts to save the situation led to the cloudy bag [4] where the chiral symmetry was restored by pions emitted from the bag surface (the pion cloud). But this model is sensitive to details of the adopted scheme of quark-pion interaction. Thus neither of the models is suited to the discussion of SCIB in finite volume.

The way to lock fermions in a finite volume without spoiling the chiral invariance is to impose the so-called **spectral boundary conditions** (SBC). They were first introduced by Atiah, Patodi and Singer (APS) who investigated anomalies on manifolds with boundaries [5]. Later these boundary conditions were widely applied in studies of index theorems on various manifolds [6].

In distinction from those mentioned before the APS conditions are non-local. They are defined on the boundary as a whole. This looks natural for finite Euclidean manifolds but is inconvenient for physical models. In the process of evolution the spatial boundary of a static physical bag becomes an infinite space-time cylinder. Constraining fields on the entire "world cylinder" would violate causality and complicate continuation to Minkowski space.

In this talk we shall demonstrate how to avoid this difficulty. One can consider a restricted spatial version of spectral boundary conditions. The modified conditions do not depend on time and are acceptable from the physical point of view. This makes possible the Hamiltonian description of the system.

The paper has the following structure. We shall review the classical APS boundary conditions in Section 1. In Section 2 we shall formulate the modified spectral conditions and discuss their properties. In conclusion we shall summarize the results and mention future prospects.

1 The physics of APS boundary conditions

1.1 Conventions

We shall start from the traditional form of SBC. First we will introduce coordinates, Dirac matrices and the gauge that allow to define the spectral boundary conditions in a convenient way.

Let us consider massless fermions interacting with gauge field $\hat{A}$ in a closed 4-dimensional Euclidean domain B_4. We choose a curvilinear coordinate frame such that in the vicinity of the boundary ∂B_4 the first coordinate ξ

points along the outward normal ($\xi = 0$ corresponds to ∂B_4) while the three others, q^i, parametrize ∂B_4 itself. For simplicity we shall assume that near the surface the metric $g_{\alpha\beta}$ depends only on q so that

$$ds^2 = d\xi^2 + g_{ik}(q)\, dq^i\, dq^k. \tag{1}$$

Moreover we choose the gauge so that on the boundary $\hat{A}_\xi = 0$.

Now we must fix the Dirac matrix γ^ξ. Let I be the 2×2 unity matrix. Then

$$\gamma^\xi = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}; \qquad \gamma^q = \begin{pmatrix} 0 & \sigma^q \\ \sigma^q & 0 \end{pmatrix}. \tag{2}$$

Matrices σ^q are the ordinary Pauli σ-matrices. With these definitions the Dirac operator of massless fermions on the surface takes the form,

$$\begin{aligned} -i\nabla\!\!\!/|_{\partial B_4} &= -i\gamma^\alpha \nabla_\alpha = \begin{pmatrix} 0 & \hat{M} \\ \hat{M}^\dagger & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & I\,\partial_\xi - i\hat{\nabla} \\ -I\,\partial_\xi - i\hat{\nabla} & 0 \end{pmatrix}, \end{aligned} \tag{3}$$

where $\hat{\nabla} = \sigma^q\, \nabla_q$ is the convolution of covariant gradient along the boundary ∇_q with σ-matrices. Note that Hermitian conjugated operators $\hat{M}$ and $\hat{M}^\dagger$ differ only by the sign of ∂_ξ-derivative.

Further on we shall call the covariant derivative $-i\hat{\nabla}$ on the boundary the **boundary operator.** It is a linear differential operator acting on 2-spinors. It is Hermitian and includes tangential gauge field $\hat{A}_q$ and the spin connection which arises from the curvature of ∂B_4.

The massless Dirac operator anticommutes with γ^5-matrix:

$$\left\{-i\nabla\!\!\!/,\, \gamma^5\right\} = 0, \qquad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{4}$$

Thus the Lagrangian of Quantum Chromodynamics is chiral invariant. In order to exploit this one needs chiral invariant boundary conditions.

1.2 The APS boundary conditions

1.2.1 The definition

Atiah, Patodi and Singer investigated spectra of Dirac operator on manifolds with boundaries. If we separate upper and lower (left and right) components of 4-spinors the corresponding eigenvalue equation for $-i\nabla\!\!\!/$ will take the form

$$-i\nabla\!\!\!/\,\psi_\Lambda = -i\nabla\!\!\!/ \begin{pmatrix} u_\Lambda \\ v_\Lambda \end{pmatrix} = \Lambda \begin{pmatrix} u_\Lambda \\ v_\Lambda \end{pmatrix} = \Lambda\,\psi_\Lambda. \tag{5}$$

The next step is to Fourier expand u and v on the boundary. Let 2-spinors $e_\lambda(q)$ be eigenfunctions of the boundary operator $-i\hat{\nabla}$:

$$-i\hat{\nabla}\,e_\lambda(q) = \lambda\,e_\lambda(q). \tag{6}$$

Note that the form of this equation and the eigenfunctions $e_\lambda(q)$ depend on gauge. It is here that the gauge condition $\hat{A}_\xi(0,\,q) = 0$ becomes important.

The operator $-i\hat{\nabla}$ is Hermitian so λ are real. The functions e_λ form an orthogonal basis. In principle $-i\hat{\nabla}$ may have zero-modes on ∂B_4 but convex manifolds are not the case.

In the vicinity of the boundary spinors u_Λ and v_Λ may be expanded in series in e_λ:

$$\begin{array}{l} u_\Lambda(\xi,\,q) = \sum_\lambda f^\lambda_\Lambda(\xi)\,e_\lambda(q), \\ f^\lambda_\Lambda(\xi) = \int_{\partial B_4} e^\dagger_\lambda(q)\,u_\Lambda(\xi,\,q)\,\sqrt{g}\,d^3q; \end{array} \tag{7a}$$

$$\begin{array}{l} v_\Lambda(\xi,\,q) = \sum_\lambda g^\lambda_\Lambda(\xi)\,e_\lambda(q), \\ g^\lambda_\Lambda(\xi) = \int_{\partial B_4} e^\dagger_\lambda(q)\,v_\Lambda(\xi,\,q)\,\sqrt{g}\,d^3q; \end{array} \tag{7b}$$

where $g = \det||g_{ik}||$ is the determinant of metric on the boundary.

The spectral boundary condition states that on the boundary, *i. e.* at $\xi = 0$

$$f^\lambda_\Lambda\Big|_{\partial B_4} = 0 \qquad \text{for} \qquad \lambda > 0; \tag{8a}$$

$$g^\lambda_\Lambda\Big|_{\partial B_4} = 0 \qquad \text{for} \qquad \lambda < 0. \tag{8b}$$

Another way to say this is to introduce integral projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ onto boundary modes with positive and negative λ:

$$\mathcal{P}^+(q,\,q') = \sum_{\lambda>0} e_\lambda(q)\,e^\dagger_\lambda(q'); \qquad \mathcal{P}^-(q,\,q') = \sum_{\lambda<0} e_\lambda(q)\,e^\dagger_\lambda(q'). \tag{9}$$

Let $\mathcal{I}$ be the unity operator on the function space spanned by e_λ. Then, obviously,

$$\mathcal{P}^+ + \mathcal{P}^- = \mathcal{I}. \tag{10}$$

If we join two-dimensional projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ into 4×4 matrix $\mathcal{P}$ the spectral boundary condition for 4-spinor ψ will look as follows:

$$\mathcal{P}\,\psi|_{\partial B_4} = \begin{pmatrix} \mathcal{P}^+ & 0 \\ 0 & \mathcal{P}^- \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\Bigg|_{\partial B_4} = 0. \tag{11}$$

The projector $\mathcal{P}$ commutes with matrix γ^5:

$$\left[\mathcal{P},\, \gamma^5\right] = 0. \tag{12}$$

Therefore boundary condition (11) by construction preserves the chiral invariance.

1.2.2 The physics

Now we shall prove that the spectral boundary conditions are acceptable and explain their physical meaning. Namely, we shall show that SBC provide Hermicity of the Dirac operator and conservation of fermions in the bag. After that we will explain the origin of requirements (8).

First let us prove that Dirac operator is Hermitian. As usually, we integrate by parts the expression

$$\int_{B_4} dV\, f^\dagger\,(-i\nabla\!\!\!/\, g) = \int_{B_4} dV\,(-i\nabla\!\!\!/\, f)^\dagger\, g + \oint_{\partial B_4} dS\, f^\dagger\,(-i\gamma^\xi) g. \tag{13}$$

Now we need to show that if f and g satisfy (8) then the last term vanishes.

Conditions (8) mean that on the boundary 4-spinors f and g may be written as: $f = \begin{pmatrix} f^- \\ f^+ \end{pmatrix}$ and $g = \begin{pmatrix} g^- \\ g^+ \end{pmatrix}$, where $f^\pm$ and $g^\pm$ include only components with positive and negative λ respectively, see (7). Rewriting the boundary term in (13) we get

$$\oint_{\partial B_4} dS\, f^\dagger\,(-i\gamma^\xi) g = \oint_{\partial B_4} dS\,\left[(f^-)^\dagger\, g^+ - (f^+)^\dagger\, g^-\right] = 0, \tag{14}$$

due to the orthogonality of eigenfunctions of the boundary operator. Thus the APS boundary conditions indeed ensure the Hermicity of Dirac operator.

In addition, relation (14) guarantees conservation of fermions in the bag. Indeed, for $f = g$ the LHS is nothing but the net fermionic current through the boundary,

$$\oint_{\partial B_4} dS\, j^\xi = -i \oint_{\partial B_4} dS\, f^\dagger \gamma^\xi f = 0. \tag{15}$$

Therefore the number of fermions is conserved and particles in the spectral bag are confined.

In order to understand the origin of SBC let us rewrite the eigenvalue condition (5) near the boundary in terms of components.

$$\begin{aligned} (\partial_\xi + \lambda)\, g^\lambda_\Lambda(\xi) &= \Lambda\, f^\lambda_\Lambda(\xi); \qquad (16a) \\ -(\partial_\xi - \lambda)\, f^\lambda_\Lambda(\xi) &= \Lambda\, g^\lambda_\Lambda(\xi). \qquad (16b) \end{aligned}$$

Depending on the sign of λ these relations reduce on the boundary either to

$$\left.\frac{\partial_\xi g^\lambda_\Lambda}{g^\lambda_\Lambda}\right|_{\xi=0} = -\lambda < 0, \qquad f^\lambda_\Lambda(0) = 0 \quad \text{at} \quad \lambda > 0; \qquad (17a)$$

or to

$$\left.\frac{\partial_\xi f^\lambda_\Lambda}{f^\lambda_\Lambda}\right|_{\xi=0} = \lambda < 0, \qquad g^\lambda_\Lambda(0) = 0 \quad \text{at} \quad \lambda < 0. \qquad (17b)$$

Thus both components either vanish on the boundary or have a negative logarithmic derivative along the normal.

This requirement has a simple physical interpretation. Suppose that out of the bag the metric and the gauge field remain the same as on the boundary. Then we can continue the functions f and g to $\xi = \infty$. Outside the bag the functions will be square integrable falling exponents as if the particle was locked in a potential well. The only difference is that now the potential for every mode is adjusted specially. We may conclude that the spectral boundary conditions claim that wave functions must have square integrable continuation to the infinite space.

2 The SBC for physical bags

2.1 The truncated SBC

Now let us turn to fermions in the infinite Euclidean cylinder $B_3 \otimes R$. We shall call the first three coordinates "space" and the fourth one "time". The boundary operator consists of spatial and temporal parts:

$$-i\hat{\nabla}_{\partial B_3 \otimes R} = -i\hat{\nabla}_{\partial B_3} - i\sigma^z \partial_4. \qquad (18)$$

We shall call the spatial part $-i\hat{\nabla}_{\partial B_3}$ the **truncated boundary operator**. Let its eigenfunctions be $e^\pm_\lambda$:

$$-i\hat{\nabla}_{\partial B_3}\, e^\pm_\lambda(q) = \pm\lambda\, e^\pm_\lambda(q), \qquad \lambda > 0. \qquad (19)$$

Wave functions on the space-time boundary $\partial B_3 \otimes R$ can be expanded in $e^{\pm}_{\lambda}$ and longitudinal plane waves:

$$u_\Lambda = \sum_{\lambda>0} \int \frac{dk}{2\pi}\, e^{ikt} \left[f_\Lambda^{+\lambda,\,k}\, e^{+}_{\lambda} + f_\Lambda^{-\lambda,\,k}\, e^{-}_{\lambda} \right]; \tag{20a}$$

$$v_\Lambda = \sum_{\lambda>0} \int \frac{dk}{2\pi}\, e^{ikt} \left[g_\Lambda^{+\lambda,\,k}\, e^{+}_{\lambda} + g_\Lambda^{-\lambda,\,k}\, e^{-}_{\lambda} \right]. \tag{20b}$$

The truncated operator $-i\hat{\nabla}_{\partial B_3}$ anticommutes with σ^z. Therefore σ^z changes the sign of e-eigenvalues. A possible choice of eigenvectors is (see [7, 8] for the sphere)

$$e^{\pm}_{\lambda} = \pm i \sigma^z\, e^{\mp}_{\lambda}. \tag{21}$$

Thus the last term in (18) mixes positive and negative spatial harmonics.

In classical approach this would mean that now SBC should be written in terms of k-dependent eigenfunctions of the full boundary operator (18). However physically these "future-sensitive" boundary conditions look strange. Therefore we propose to apply independent of k truncated APS constraints:

$$\left. f_\Lambda^{+\lambda,\,k} \right|_{\partial B_3} = 0; \tag{22a}$$

$$\left. g_\Lambda^{-\lambda,\,k} \right|_{\partial B_3} = 0. \tag{22b}$$

These conditions do not depend on time and allow Hamiltonian treatment of the system. Moreover, they may be applied both in Euclidean and Minkowski spaces. Now let us show that they are acceptable.

2.2 Consistency

We are going to prove that the truncated form of SBC fulfills necessary conditions. Namely that they are chiral invariant, that the Dirac operator is Hermitian and the fermionic current is conserved and after all that wave functions may be continued out of the bag.

The proof of the first three points literally follows the 4-dimensional case. Everything that concerns formulae (9–15) remains true for truncated ($_T$) 3-dimensional SBC (22). One may define on ∂B_3 projectors,

$$\mathcal{P}^{\pm}_{T}(q,\, q') = \sum_{\lambda>0} e^{\pm}_{\lambda}(q)\, \left[e^{\pm}_{\lambda}(q') \right]^{\dagger}. \tag{23}$$

Then the truncated boundary conditions may be written in the manifestly γ^5-invariant form,

$$\mathcal{P}_T\,\psi|_{\partial B_3} = \left(\begin{array}{cc} \mathcal{P}_T^+ & 0 \\ 0 & \mathcal{P}_T^- \end{array} \right) \left(\begin{array}{c} u \\ v \end{array} \right) \Bigg|_{\partial B_3} = 0. \tag{24}$$

Hermicity of the Dirac operator and conservation of fermions are proven in the same way as before, see (13–15). We don't rewrite the formula.

The last point is more delicate. We already mentioned that the σ^z term in (18) mixes positive and negative harmonics. Therefore they must be analysed together and instead of two eigenvalue equations (16) we get four (ξ is the normal to the spatial boundary):

$$\begin{aligned} (\partial_\xi + \lambda)\, g_\Lambda^{+\lambda,\,k} &= \Lambda\, f_\Lambda^{+\lambda,\,k} + ik\, g_\Lambda^{-\lambda,\,k}; && \text{(25a)} \\ -(\partial_\xi - \lambda)\, f_\Lambda^{+\lambda,\,k} &= \Lambda\, g_\Lambda^{+\lambda,\,k} + ik\, f_\Lambda^{-\lambda,\,k} : && \text{(25b)} \\ (\partial_\xi - \lambda)\, g_\Lambda^{-\lambda,\,k} &= \Lambda\, f_\Lambda^{-\lambda,\,k} - ik\, g_\Lambda^{+\lambda,\,k}; && \text{(25c)} \\ -(\partial_\xi + \lambda)\, f_\Lambda^{-\lambda,\,k} &= \Lambda\, g_\Lambda^{-\lambda,\,k} - ik\, f_\Lambda^{+\lambda,\,k}. && \text{(25d)} \end{aligned}$$

The new feature with respect to (16) are ik addends that appear due to the mixing. However one may notice that the terms in the RHS of (25) come in pairs f^+, g^- and f^-, g^+. Therefore according to conditions (22) the RHS of equations (25a, 25d) vanish on the boundary. Thus the behaviour of g^+ and f^- on the boundary is governed by the homogeneous equations and

$$\left.\frac{\partial_\xi f_\Lambda^{-\lambda,\,k}}{f_\Lambda^{-\lambda,\,k}}\right|_{\xi=0} = \left.\frac{\partial_\xi g_\Lambda^{+\lambda,\,k}}{g_\Lambda^{+\lambda,\,k}}\right|_{\xi=0} = -\lambda < 0. \tag{26}$$

Hence despite the presence of extra pieces the nonvanishing components g^+ and f^- have negative logarithmic derivatives. This means that solutions of eigenvalue equations may be continued outwards of the "world cylinder" in an integrable way and the last of our requirements is fulfilled. This completes the proof of acceptability of the truncated SBC.

Conclusion

The truncated version of APS boundary conditions offers a number of possibilities. It allows to formulate a chiral invariant bag model and to approach chiral properties of fermionic field in the closed volume. The constraints are imposed on the spatial boundary of the bag so one may write down the Hamiltonian and study the energy spectrum of the system. Another advantage is that the modified SBC do not depend on time and may be used both in Euclidean and Minkowsky space.

A new feature that SBC may bring to the bag physics is their nonlocality. Other bag models [1, 2, 3] employed local boundary conditions which correspond to the thin wall approximation. The nonlocal spectral conditions refer to the boundary as a whole. Therefore in a sense hadrons are also treated as a whole. This looks promising by itself. To begin with it would be interesting to investigate hadronic spectra in chiral invariant bags. This could answer whether the model is realistic and indicate missing elements.

Another question is more mathematical. Chiral symmetry is a specific of fermions in even dimensional spaces. Hence the spectral boundary conditions were also considered in even dimensions. The truncated SBC are formulated in the odd dimensional space that remains after discarding the time. This might have interesting consequences. For example, the boundary of odd-dimensional bag is an even-dimensional manyfold and one can introduce a sort of internal chirality for surface modes. The question is if there is a way for this hidden symmetry to reveal itself.

In conclusion I would like to express my gratitude to Professor A. Wipf for elucidating discussions. I thank the Organizers for financial support. The work was supported by RFBR grant 03–02–16209 and Federal Program of the Russian Ministry of Industry, Science and Technology No 40.052.1.1.1112.

References

[1] A. Chodos, R. L. Jaffe, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. **D9** (1974) 3471.

[2] S. Duerr, A. Wipf, Nucl. Phys. **B443** (1995) 201.

[3] G. Esposito, K. Kirsten, Phys. Rev. **D66** (2002) 085014.

[4] S. Théberge, A. W. Thomas, G. A. Miller, Phys. Rev. **D22** (1980) 2838; Phys. Rev. **D23** (1981) 2106(E).

[5] M. F. Atiah, V. K. Patodi, I. M. Singer, Math. Proc. Camb. Phil. Soc. **77** (1975) 43.

[6] T. Eguchi, P. B. Gilkey, A. J. Hanson, Phys. Rep. **C 66** (1980) 213.

[7] A. A. Abrikosov, Int. Journ. of Mod. Phys. **A17** (2002) 885.

[8] A. A. Abrikosov, hep-th/0212134.