Parquet Approximation for Two–Matrix Model

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Abstract

The parquet planar approach is reviewed and applied to the zero-dimensional two-matrix model.

As it can be seen from the title, the parquet planar approximation is constructed on the base of two different approaches. To begin I would like to recall some general ideas that underline both of them.

The parquet approximation or generalized ladder approximation was proposed by Landau, Khalatnikov, Abrikosov, as a tool for describing QED [1]. The parquet approximation can be defined as a solution of a certain closed set of integro-differential equations on propagators and vertex functions, and all other structures of the theory (e.g. Green functions) are constructed in terms of this solution. The main characteristic feature of this set is that it has sense for both small and large values of the coupling constant. Later on, this approximation was applied to different models (meson–meson scattering, four-fermion interaction) [2].

The planar approximation is based on the observation that in some matrix field theories, e.g. the gauge theory with $SU(N)$ gauge group, the perturbation theory can be constructed with $1/N$ as a small parameter, and in the

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limit of large N the main contribution is the diagrams that can be depicted on the plane without self-intersection [3].

Both approaches have advantages and disadvantages. For example, the parquet approximation, unlike the planar one, violates the gauge symmetry preventing so its direct application for gauge theories. At the same time the planar approximation is very difficult to analyze in spaces with large dimensions, while the parquet approximation is independent of the dimensionality of the space.

So the hope when constructing the parquet planar approximation is to propose a method that would combine pro and take into account contra of both approaches.

1. The parquet planar approximation in its first form was proposed in [5]. They considered a so-called hermitian one matrix model with the action

$$S(M) = \frac{1}{2}\mathrm{Tr}\, M^2 + \frac{g}{4N}\mathrm{Tr}\, M^4.$$

Note that the theory is considered in the zero-dimensional space-time.

This model, as all polynomial matrix models, has deep connections with many problems in physics and mathematics (2D gravity, exactly solvable models).

Defining the Green functions as follows

$$\Pi_n = <\mathrm{Tr}\, M^n> \equiv \lim_{N\to\infty} \mathcal{N}\frac{1}{N^{1+n/2}}\int DM \mathrm{Tr} M^n\, e^{-S(M)},$$

one can show that they satisfy the planar Schwinger–Dyson equations of the form

$$\Pi_n + g\Pi_{n+2} = \sum_{i=0}^{n-2}\Pi_i\Pi_{n-i-2}, \qquad n \geq 2.$$

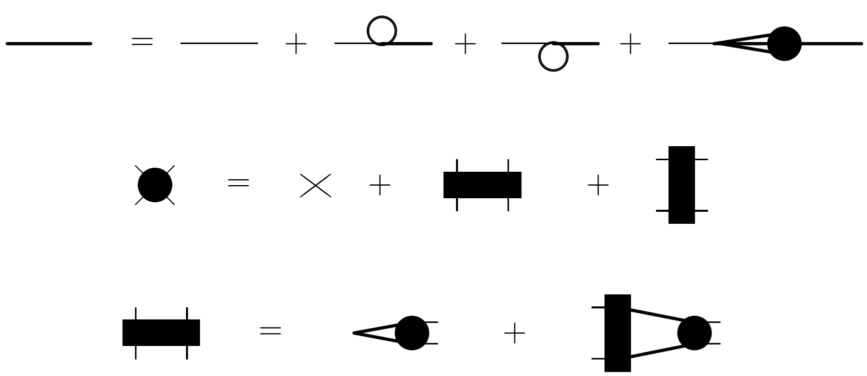
They form an infinite chain, and it can easily be shown that the whole set of the Green functions is derived in terms of the generating functional that depends on an additional free parameter. As such a parameter one can take, for example, Π_2 or Π_4, therefore, to define it one should have one additional condition. Hence, there exists a certain way to control approximations.

The parquet planar approximation is defined as a solution of the following system

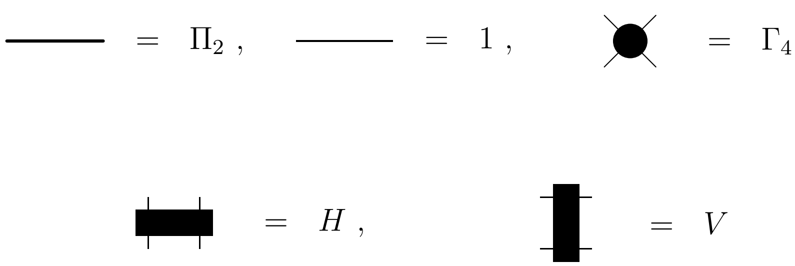
$$\begin{cases} \Pi_2 = 1 - 2g\Pi_2^2 - g\Pi_2^4\Gamma_4 = 1 - g\Pi_4 \\ \Gamma_4 = -g + H + V \\ H = -g\Pi_2^2\Gamma_4 + V\Pi_2^2\Gamma_4 \\ V = -g\Pi_2^2\Gamma_4 + H\Pi_2^2\Gamma_4 \end{cases} \tag{1}$$

here Γ_4 is the four-point vertex, H (V) is the part of the four-point vertex function that contains diagrams 2PR in the t-channel (s-channel) and not 2PR in the s-channel (t-channel). The vertices V and H are related by the cyclic permutation of external points. The first equation of (1) is the Schwinger–Dyson equation, the others are called parquet. So one can represent the parquet planar approximation as a way to close or cut the infinite Schwinger–Dyson chain and to write down the aforementioned additional condition.

The equations (1) can be represented graphically.



Here notations are as follows



The thick and thin lines represent the full and bare within the planar parquet approximation propagators respectively.

From the figures it can be seen why the method is called parquet planar: it is called parquet since diagrams that are taken into account form a parquet, and planar since non-planar contributions from the u-channel are omitted.

In the zero-dimensional case this system is purely algebraic, and its solution reproduces with perfect precision the exact planar results [4] as it is seen from the table below.

	Planar	Parquet Planar
$\Pi_2(g)$ at small g	$1-8g+o(g)$	$1-8g+o(g)$
$\Pi_2(g)$ at large g	$0.7698g^{-1/2}$	$0.7695g^{-1/2}$
g_{critical}	-0.083	-0.084

2. The next important class of hermitian matrix models are two-matrix models [7, 8]. The simplest model is the model with the action

$$S(M_1,M_2)=\mathrm{Tr}\,M_1^2+\frac{g}{N}\mathrm{Tr}\,M_1^4+\mathrm{Tr}\,M_2^2+\frac{g}{N}\mathrm{Tr}\,M_2^4-2c\mathrm{Tr}\,M_1M_2.$$

This model in the large N limit together with all mentioned above has much in common with the Ising model on random lattices [6]. The more subtle consequence of the two-matrix model is that this technique can be used to construct the 1D matrix model or matrix quantum mechanics in an indirect way.

The partition function is defined in the usual way

$$Z=\int DM_1\,DM_2\,e^{-S(M_1,M_2)}$$
$$=\mathrm{const}\int\prod_{i=1}^{N}dx_idy_i\Delta(x_i)\Delta(y_i)e^{-S(x_i,y_i)},$$
$$\Delta(x_i)=\prod_{i<j=1}^{N}(x_i-x_j)^2,$$

here M_1 and M_2 have been diagonalized to $X=\|x_i\|$ and $Y=\|y_i\|$.

The usual way to investigate matrix models, especially useful in the case of the two-matrix model, is the orthogonal polynomials technique [8]. One introduces the orthogonal or biorthogonal polynomials $P_i(x)$

$$\int\limits_{-\infty}^{\infty}P_i(x)P_j(y)e^{-S(x,y)}\,dxdy=h_i\delta_{ij},$$
$$S(x,y)=x^2+\frac{gx^4}{N}+y^2+\frac{gy^4}{N}-2cxy.$$

These polynomials satisfy the recursion formula

$$xP_i(x)=P_{i+1}(x)+R_iP_{i-1}(x)+T_iP_{i-3}(x),$$

whence it is possible to derive the set of equations on R_i, T_i, and $f_i \equiv h_i/h_{i-1}$

$$\left\{\begin{aligned} f_i &= \frac{cR_i}{1+\frac{2g}{N}(R_{i+1}+R_i+R_{i-1})} \\ cf_i &= \ -i/2 + R_i\left(1+\frac{2g}{N}(R_{i+1}+R_i+R_{i-1})\right) \\ &\quad +\frac{2g}{N}(T_{i+2}+T_{i+1}+T_i) \\ cT_i &= \frac{2g}{N} f_i f_{i-1} f_{i-2} \end{aligned}\right. \tag{2}$$

In the large N limit one introduces continuous functions

$$\frac{i}{N} \to x \in [0,1], \quad \frac{f_i}{N} \to f(x), \quad \frac{R_i}{N} \to R(x), \quad \frac{T_i}{N^2} \to T(x),$$

while the set (2) turns into

$$\left\{\begin{array}{l} f(x) = cR(x)(1+6gR(x))^{-1} \\ cf(x) = -x/2 + R(x)(1+6gR(x)) + 6gT(x) \\ cT(x) = 2gf^3(x) \end{array}\right.$$

Now the partition function will be

$$\ln Z = \ln\left(\text{const}\prod_{i=0}^{N-1} h_i\right) \to \text{const} + \int_0^1 (1-x)\ln f(x)dx.$$

One can calculate the 2-point correlation function $D_1 = < M_1^2 > \equiv < M_2^2 > = D_2$

$$D_1 = \lim_{N\to\infty} \mathcal{N} \int dM_1\, dM_2\, \mathrm{Tr}\, M_1^2 \exp(-S(M_1, M_2))$$
$$= f(0) + 2\int_0^1 (1-x)f'(x)dx.$$

Then its asymptotic behavior is defined by the following expression

$$D_1 = \frac{1}{2(1-c^2)} - \frac{g(1+c^2)}{(1-c^2)^3} + \frac{g^2(3c^2+9)(2c^2+1)}{2(1-c^2)^5} + O(g^3). \tag{3}$$

3. The parquet planar approximation for the two-matrix model is defined as a solution of the following set of equations.

The first four equations are

$$\begin{cases} D_1 = 1 - 8gD_1^2 - 4gD_1^4\Gamma_1 + c\Pi_2 \\ D_2 = 1 - 8gD_2^2 - 4gD_2^4\Gamma_2 + c\Pi_2 \\ \Pi_2 = cD_1 - 8gD_2\Pi_2 \\ \Pi_2 = cD_2 - 8gD_1\Pi_2 \end{cases} \tag{4}$$

These equations are the Schwinger–Dyson equations on both propagators $D_1 =< \operatorname{Tr} M_1^2 >$, $D_2 =< \operatorname{Tr} M_2^2 >$ and the effective vertex $\Pi_2 =< \operatorname{Tr} M_1 M_2 >$ that describes the transformation of matrix fields one into another; Γ_1 and Γ_2 are four-point vertices for both matrix fields; by definition mixed vertices are not considered. The vertices Γ_1 and Γ_2 are defined by the following parquet planar equations

$$\begin{cases} \Gamma_1 = -4g + H_1 + V_1 \\ H_1 = -4gD_1^2\Gamma_1 + V_1D_1^2\Gamma_1 \\ V_1 = -4gD_1^2\Gamma_1 + H_1D_1^2\Gamma_1 \\ \Gamma_2 = -4g + H_2 + V_2 \\ H_2 = -4gD_2^2\Gamma_2 + V_2D_2^2\Gamma_2 \\ V_2 = -4gD_2^2\Gamma_2 + H_2D_2^2\Gamma_2 \end{cases} \tag{5}$$

This set (4) and (5) can be solved in limit of small g. The asymptotic expression for the two-point Green function is

$$D_{1,2} = \frac{1}{2(1-c^2)} - \frac{g(1+c^2)}{(1-c^2)^3} + \frac{g^2(2c^2+1)(2c^2+9)}{2(1-c^2)^5} + O(g^3), \tag{6}$$

and it behaves similarly to the planar case (3).

It is possible to derive an equation solely on D_1, g, and c. This equation of the 8th order in D_1 defines D_1 as the function of the coupling constants. Its graph has two sheets in the physically interesting domain, that is $0 \le c \le 1$. These two sheets correspond to different phases of the model, while the transition from one sheet to another is associated with a phase transition.

Note that setting c equal to zero, i.e. when there are two non-interacting theories, one gets the usual solution for the one-matrix model considered in [5].

4. To conclude, let us list possible consequences and generalizations.

First, this model admits direct generalization for the case of several matrices with chain interaction $\sum_{i=1}^{p-1} c_i \mathrm{Tr}\, M_i M_{i+1}$, so-called multi-matrix models [9]. It is interesting to investigate its solution or at least the behavior of this solution in the limit $p \to \infty$. It should give the matrix quantum mechanics [10].

The second point is that so far the parquet approximation has been applied to the planar or spherical limit, i.e. the large N limit. It is interesting to know if it is possible to consider cases with general topology. In other words, the question is whether it is possible to formulate the parquet approximation in the double-scaling limit.

It is necessary to mention that the parquet approach does not contain any parameter that could control this approximation[1]. Hence, one should not pretend to get the complete solution using the parquet approximation, it is merely a tool for investigating some properties (e.g. critical constants, the behavior of solutions).

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