# Harmonic Maps and Gravitating Monopoles and Skyrmions *

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**Abstract**

We present some results of our recent work [1] on the solutions of the $SU(N)$ Einstein-Skyrme system. In this work we expressed the chiral fields (which were not simple embeddings of the $SU(2)$ one) in terms of 2 dimensional $S^2 \to S^2$ harmonic maps. This has allowed us

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to derive the $SU(N)$ spherical symmetric equations for any $N$ and to study various gravitating skyrmion solutions of these equations.

In particular, we focus our attention on the $SU(3)$ case and show that it possesses three different types of gravitating skyrmions with topological charge 4, 2 and 0. We discuss some properties of these solutions.

# 1 Introduction

In this talk we present some results obtained by us when studying classical equations of the $SU(N)$ Einstein - Yang Mills and Skyrme systems [1, 2].

In each case we looked at their classical equations and tried to find their solutions, or their approximate solutions, by using the 2-dimensional Harmonic Maps [3].

For the $SU(N)$ Einstein - Skyrme system the action reads:

$$S = \int \left[ \frac{R}{16\pi G} - \frac{1}{2}\,\mathrm{tr}\left(K_\mu\, K^\mu\right) - \frac{1}{16}\,\mathrm{tr}\left(\left[K_\mu, K_\nu\right]\left[K^\mu, K^\nu\right]\right)\right] \sqrt{-g}\, d^4x\ . \quad (1)$$

Here $K_\mu = \partial_\mu U U^{-1}$ for $\mu = 0, 1, 2, 3$ and $U(x^\mu) \in SU(N)$ is the matter field. Moreover, $g$ denotes the determinant of the metric and $G$ represents Newton's constant.

And for the $SU(N)$ Einstein - Yang Mills - Higgs system the action is given by:

$$\begin{aligned} S &= \int \left[ \frac{R}{16\pi G} - \frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}\, F^{\mu\nu}\right) - \frac{1}{4}\mathrm{tr}\left(D_\mu\Phi\, D^\mu\Phi\right)\right]\sqrt{-g}d^4x \\ &- \frac{1}{8}\int \left[\lambda\left(\mathrm{tr}\left(\Phi^2 - \eta^2\right)\right)^2\right]\sqrt{-g}\, d^4x\ . \end{aligned} \quad (2)$$

Here

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (3)$$

and the covariant derivative of the Higgs field $\Phi$ is given by:

$$D_\mu\Phi = \partial_\mu\Phi + [A_\mu, \Phi]\ . \quad (4)$$

The matrix $\eta$ represents $\eta = iv\mathbf{1}_N$, $v \in R$, where $\mathbf{1}_N$ denotes the unit matrix in $N$ dimensions. $\lambda$ is the Higgs self-coupling constant and $v$ the vacuum expectation value of the Higgs field.

In this talk we show how to find static solutions of the coupled system of Einstein- and Euler Lagrange equations corresponding to these Lagrangians. In this construction we use the 2-dimensional Harmonic Maps. Because of the lack of time the discussion is given only for the Einstein-Skyrme system but it generalises very simply to the case of monopoles [2].

Before we discuss the solutions of the Einstein-Skyrme model let us add that studying this model is not a purely mathematical excercise but that this model can, perhaps, be used to describe some real systems; ie it could be used in the description of the interaction between a baryon and a black hole (a configuration which might have been produced in the very early universe). It may also be relevant, for large baryon number, for the description of neutron stars etc.

Let us add also that, so far, most of the studies of the gravitational skyrmions, have concentrated on the $SU(2)$ Einstein-Skyrme model [4, 5, 6]. In particular, [4] showed that the Schwarzschild black hole could support chiral ("Skyrme") hair and argued that such configurations might be stable. The presence of the horizon in the core of the skyrmion can unwind the skyrmion, leaving fractional baryon charge outside the horizon.

Further investigations of the model were undertaken in [5]. In addition, globally regular solutions with baryon number one [6] and black holes with chiral hair were found and their stability properties were studied in detail [5, 6].

The first examples of nonembedded solutions for a higher group, namely the $SU(3)$ group, were the $SO(3)$ solitons with even topological charge. It was shown that the lowest energy solution corresponded to a bound state of two gravitating skyrmions [7]. Specifically, it was found that there were two branches of these regular solutions and that these branches merged at a critical value of the gravitational coupling.

In our work [1] we considered particle-like solutions of the $SU(N)$ Einstein-Skyrme model (for $N \geq 2$). In particular, we studied the deformation of the pure multiskyrmion configurations [8] (derived using the harmonic map ansatz) when gravity was introduced. New types of solutions were found, which corresponded to skyrmion-antiskyrmion configurations with topological charge 0. Like the non-gravitating skyrmion-antiskyrmions, these configurations are also saddle points of the energy functional and thus are unstable.

## 2 Equations for $SU(N)$ Einstein-Skyrme Model

To derive the classical equations of motion of (1) we performed the variation of the action (1) with respect to the metric and the Skyrme field. The variation with respect to the metric $g^{\mu\nu}$ gave us the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}, \tag{5}$$

where $R_{\mu\nu}$ denotes the Ricci tensor and the stress-energy tensor $T_{\mu\nu} = g_{\mu\nu}\mathcal{L} - 2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}$ is given by:

$$T_{\mu\nu} = \mathrm{tr}\Big(K_\mu K_\nu - \frac{1}{2}g_{\mu\nu}K_\alpha K^\alpha\Big) + \frac{1}{4}\mathrm{tr}\Big(g^{\alpha\beta}\,[K_\mu, K_\alpha]\,[K_\nu, K_\beta] - \frac{1}{4}g_{\mu\nu}\,[K_\alpha, K_\beta]\,\big[K^\alpha, K^\beta\big]\Big). \tag{6}$$

The variation with respect to the matter fields leads to the Euler-Lagrange equations which will be discussed in the next sections.

Note that the Einstein-Skyrme system has a topological current which is covariantly conserved, yielding the topological charge [9]:

$$B = \int \sqrt{-g}\, B^0\, d^3x \quad , \quad B^\mu = -\frac{1}{24\pi^2\sqrt{-g}}\,\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\left(K_\nu K_\alpha K_\beta\right) \tag{7}$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the (constant) fully antisymmetric tensor.

## 3 Harmonic Map Ansatz

First we change our variables

$$(x, y, z) \quad \rightarrow \quad (r, \theta, \phi) \quad \rightarrow \quad (r, z, \bar{z}), \tag{8}$$

where the Riemann sphere variable $z$ is given by $z = e^{i\phi}\tan(\theta/2)$ and $\bar{z}$ is the complex conjugate of $z$. For the metric we take

$$ds^2 = -A^2(r)C(r)dt^2 + \frac{1}{C(r)}dr^2 + \frac{4r^2}{(1+|z|^2)^2}\,dzd\bar{z}\ , \tag{9}$$

$$C(r) = 1 - \frac{2m(r)}{r}\ . \tag{10}$$

Then

$$\sqrt{-g} = iA(r)\,\frac{2r^2}{(1+|z|^2)^2}\ . \qquad (11)$$

For the $U$ fields we put

$$\begin{aligned} U &= \exp\left\{2i\sum_{i=0}^{N-2} g_i\left(P_i - \frac{I}{N}\right)\right\} \\ &= e^{-2ig_0/N}\,(1+A_0P_0)\,e^{-2ig_1/N}\,(1+A_1P_1)\times .. \\ &\qquad \times e^{-2ig_{N-2}/N}\,(1+A_{N-1}P_{N-2})\ , \end{aligned} \qquad (12)$$

where $g_k = g_k(r)$ for $k = 0,\ldots,N-2$ are the profile functions which depend only on $r$. Moreover, we define also $A_k = e^{2ig_k} - 1$. The boundary value $U \to I$ at $r \to \infty$ (needed for finiteness of the action) imposes the requirement that $g_i(\infty) = 0$.

Here $P_j(z,\bar{z})$ are $N \times N$ Hermitian projectors: $P_j = P_j^\dagger = P_j^2$, (and are independent of the radius $r$). Projectors $P_k$ are defined by [10]:

$$P_k = \frac{(\Delta^k f)^\dagger \Delta^k f}{|\Delta^k f|^2}\ , \qquad k = 0,..,N-1, \qquad (13)$$

where

$$\Delta f = \partial_z f - \frac{f\,(f^\dagger\,\partial_z f)}{|f|^2}\ . \qquad (14)$$

($\Delta^k f$ is defined via $\Delta(\Delta^{k-1} f)$.

These projectors form a set of **harmonic maps** (of 2 dim $CP^N$ model) - hence their name. The initial holomorphic vector $f$ is taken as

$$f = (f_0, ..., f_j, ..., f_{N-1})^t, \quad \text{where} \quad f_j = z^j \sqrt{\binom{N-1}{j}} \qquad (15)$$

and $\binom{N-1}{j}$ denote the binomial coefficients.

Note that with this choice of $f$ the equations for the $U$ fields reduce to the equations for the profile functions (this is true irrespectively whether the gravitational field is coupled in or not).

# 4 Spherical Symmetric equations of motion

In the case of spherical symmetry, as we have shown in [1], the action takes the form

$$\begin{aligned} S &= 2\pi \int \Bigg\{ \frac{RAr^2}{8\pi G} - \frac{4}{N} r^2 CA \left( \sum_{i=0}^{N-2} g_i' \right)^2 + 4r^2 CA \sum_{i=0}^{N-2} g_i'^2 \\ &\quad + 2A \sum_{k=1}^{N-1} D_k + \frac{A}{4r^2} \left[ D_1^2 + \sum_{i=1}^{N-2} (D_i - D_{i+1})^2 + D_{N-1}^2 \right] \\ &\quad + 2CA \sum_{k=1}^{N-1} D_k \left( g_k' - g_{k-1}' \right)^2 \Bigg\} drdt. \end{aligned} \tag{16}$$

Here $D_k = 2k(N-k)\sin^2(g_k - g_{k-1})$. The matter equations are obtained from the variation of this action with respect to the matter field. We will discuss these equations for the specific case of $SU(3)$.

In addition, the Einstein equations take the form:

$$\begin{aligned} \frac{2}{r^2} m' &= 16\pi G \Bigg[ - \frac{C}{N} \left( \sum_{i=0}^{N-2} g_i' \right)^2 + C \sum_{i=0}^{N-2} g_i'^2 + \frac{1}{2r^2} \sum_{k=1}^{N-1} D_k \\ &\quad + \frac{C}{2r^2} \sum_{k=1}^{N-1} D_k \left( g_k' - g_{k-1}' \right)^2 \\ &\quad + \frac{1}{16r^4} \left( D_1^2 + \sum_{i=1}^{N-2} (D_i - D_{i+1})^2 + D_{N-1}^2 \right) \Bigg], \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{2}{r} \frac{A'}{A} C &= 16\pi G \Bigg[ - \frac{2C}{N} \left( \sum_{i=0}^{N-2} g_i' \right)^2 + 2C \sum_{i=0}^{N-2} g_i'^2 \\ &\quad + C \frac{1}{r^2} \sum_{k=1}^{N-1} D_k \left( g_k' - g_{k-1}' \right)^2 \Bigg]. \end{aligned} \tag{18}$$

For simplicity, we set $F_k = g_k - g_{k+1}$ for $k = 0, \ldots, N-2$ with $F_{N-2} = g_{N-2}$. Moreover, the topological charge is now given by

$$B = \frac{1}{\pi} \sum_{i=0}^{N-2} (i+1)(N-i-1) \left( F_i - \frac{\sin 2F_i}{2} \right) \Bigg|_0^\infty . \tag{19}$$

Since $F_i(\infty) = 0$ the only contributions to the topological charge come from $F_i(0)$.

### 4.1 The case of $SU(3)$

For $N = 3$ there are two profile functions, $F_0(r)$, $F_1(r)$, and (16) becomes

$$\begin{aligned} S &= 2\pi \int \left\{ \frac{RAr^2}{8\pi G} + \frac{8}{3} r^2 CA \left( F_0'^2 + F_1'^2 + F_0' F_1' \right) \right. \\ &\quad +8A \left( \sin^2 F_0 + \sin^2 F_1 \right) \\ &\quad + \frac{8A}{r^2} \left( \sin^4 F_0 - \sin^2 F_0 \sin^2 F_1 + \sin^4 F_1 \right) & (20) \\ &\quad \left. +8CA \left( \sin^2 F_0 F_0'^2 + \sin^2 F_1 F_1'^2 \right) \right\} drdt \; . & (21) \end{aligned}$$

The corresponding equations for $F_0$ and $F_1$ are now given by:

$$\left[ r^2 CA F_0' \left( \frac{2}{3} + \frac{2\sin^2 F_0}{r^2} \right) + \frac{r^2}{3} CA\, F_1' \right]' - A \sin 2F_0 \left( 1 + \frac{2\sin^2 F_0 - \sin^2 F_1}{r^2} + CF_0'^2 \right) = 0 \; , \quad (22)$$

$$\left[ r^2 CA F_1' \left( \frac{2}{3} + \frac{2\sin^2 F_1}{r^2} \right) + \frac{r^2}{3} CA\, F_0' \right]' - A \sin 2F_1 \left( 1 + \frac{2\sin^2 F_1 - \sin^2 F_0}{r^2} + CF_1'^2 \right) = 0 \; . \quad (23)$$

Note that the above equations are symmetric under the simultaneous interchange $F_0 \rightarrow F_1$ and $F_1 \rightarrow F_0$.

Finally, the Einstein equations (5) take the form:

$$\begin{aligned} \frac{2}{r^2} m' &= 32\pi G \left[ \frac{C}{3} \left( F_0'^2 + F_0' F_1' + F_1'^2 \right) + \frac{1}{r^2} \left( \sin^2 F_0 + \sin^2 F_1 \right) \right. \\ &+ \frac{C}{r^2} \sin^2 F_0 F_0'^2 + \frac{C}{r^2} \sin^2 F_1 F_1'^2 \\ &+ \left. \frac{1}{r^4} \left( \sin^4 F_0 - \sin^2 F_0 \sin^2 F_1 + \sin^2 F_1 \right) \right] & (24) \end{aligned}$$

$$\frac{2}{r}\frac{A'}{A} = 64\pi\, G \left[\frac{1}{3}\left(F_0'^2 + F_0'F_1' + F_1'^2\right) + \frac{\sin^2 F_0}{r^2}F_0'^2 + \frac{\sin^2 F_1}{r^2}F_1'^2\right] . \qquad (25)$$

The set of equations (22)-(25) can only be solved numerically when the right boundary conditions have been imposed. Similarly to the flat case [8], we see that there exist three types of gravitating multiskyrmions which we will discuss in detail in the following section.

# 5 Numerical Results

To solve the equations (22)-(25) numerically, we have adopted the numerical routine described in [11]. For convenience, we define $\alpha^2 = 16\pi G$ so that the flat limit with $C(r) = A(r) = 1$ corresponds to $\alpha = 0$.

## 5.1 Boundary conditions

Clearly, to have regular finite-energy solutions we can impose the following boundary conditions:

$$\begin{array}{rll} \text{(I)} & F_0(0) = \pi, & F_1(0) = \pi, \\ \text{(II)} & F_0(0) = \pi, & F_1(0) = -\pi, \\ \text{(III)} & F_0(0) = \pi, & F_1(0) = 0, \end{array} \qquad (26)$$

together with

$$\text{(I) / (II) / (III)} \;:\; F_0(\infty) = 0 \;\;,\;\; F_1(\infty) = 0 \; . \qquad (27)$$

We also have two supplementary conditions for the metric functions:

$$\text{(I) / (II) / (III)} \;:\; m(0) = 0 \;\;,\;\; A(\infty) = 1 \; . \qquad (28)$$

The condition that $m(0) = 0$ vanishes guarantees regularity, while the condition on $A(r)$ comes from the requirement of asymptotic flatness. The energy $E$ of the gravitating skyrmions can then be determined from the "mass function" $m(r)$ at infinity:

$$E = \frac{4m(\infty)}{3\pi\alpha^2} \; . \qquad (29)$$

With this normalisation, the values of $E$ can be compared to those of the flat limit [8].

Next we present the results of our numerical analysis which demonstrate that the three solutions are indeed continuously deformed by gravity ($\alpha > 0$).

## 5.2 Case I

This case corresponds to choosing $F_0(r) = F_1(r)$. In this case the equations, after the rescaling, coincide with those of the gravitating $SU(2)$ skyrmion, which were studied in [5, 6]. For completeness, we present here their main features.

The **non-gravitational** solution has energy $E \approx 4.928 = 4 \times 1.232$, i.e. four times the energy of the $SU(2)$ one-skyrmion. Due to the boundary conditions $F_0(0) = F_1(0) = \pi$ its topological charge is four. We can interpret this non-gravitating solution as describing our noninteracting skyrmions placed on top of each other in such a way that the baryon (energy) density is spherically symmetric.

In the **gravitational** case, from the boundary conditions, we have

$$
\begin{aligned}
F_0(r) = F_1(r) &\approx \pi - B_I r \qquad \text{for} \qquad 0 \le r << 1 \;\; , \\
F_0(r) = F_1(r) &\approx \frac{\tilde{B}_I}{r^2} \qquad \text{for} \qquad r >> 1 \; ,
\end{aligned}
\tag{30}
$$

where $B_I$, $\tilde{B}_I$ are (shooting) parameters depending on $\alpha$ which have to be determined numerically.

Solving numerically our equations we have found that the flat solution is deformed by gravity. Thus $C(r)$ develops a local minimum at some intermediate radius : $r = r_m(\alpha)$, while the function $A(r)$ has a minimum $A_{min} = A(0)$ at the origin and then increases monotonically. The Skyrme field profile functions deviate only slightly from the corresponding ones in the flat limit. As $\alpha$ increases, the respective minimal values of $C(r)$, $A(r)$, i.e. $C_m = C(r_m)$ and $A_0 = A(0)$ both decrease and so does the corresponding energy $E$. We can interpret this as the gravitational binding of 4 skyrmions.

Note that the branch of gravitating skyrmions exists only up to some critical value $\alpha_{cr}$: $\alpha \le \alpha_{cr} \approx 0.142087$. Note also that $A_0$, $C_m$, as functions of $\alpha$, remain finite with

$$
\begin{aligned}
&E(\alpha = \alpha_{cr}) \approx 4.20, \\
&A_0(\alpha = \alpha_{cr}) \approx 0.437, \\
&C_m(\alpha = \alpha_{cr}) \approx 0.584.
\end{aligned}
\tag{31}
$$

We have found also **a second branch** of solutions in the interval $[0, \alpha_{cr}]$. For a given $\alpha$, the second branch solution has a higher energy while $A_0$, $C_m$ have lower values. For $\alpha \to \alpha_{cr}$ both branches go to the same solution. For

$\alpha \to 0$ the second branch becomes more and more peaked around the origin (i.e. the slope $F'(0)$ tends to infinity) and its energy diverges as $\alpha \to 0$ (but the product $\alpha E$ remains finite ($\approx 0.124$)). This solution, rescaled according to $x \to \frac{x}{\alpha}$, stays regular in the $\alpha \to 0$ limit and converges to a $SU(3)$ sphaleron.

Note that the metric functions remain finite and are strictly positive. Thus no black hole solution is generated by the solutions of the equations under consideration, in contrast to e.g. the Einstein-Yang-Mills-Higgs equations.

## 5.3 Case II

Put $F_0(r) = -F_1(r)$ with $F_0(0) = -F_1(0) = \pi$. Then the **non-gravitating** solution is topologically trivial since the topological charge is zero. However, it is not the vacuum solution but it describes a system of two skyrmions and two antiskyrmions.

Note that the conditions on $F_0(r)$ are now:

$$
\begin{aligned}
F_0(r) &\approx \pi - B_{II} r^2 \qquad \text{for} \qquad 0 \le r << 1 \ , \\
F_0(r) &\approx \frac{\tilde{B}_{II}}{r^3} \qquad \text{for} \qquad r >> 1 \ .
\end{aligned}
$$

We have then checked the regularity of the solution which implies then $F_0'(0) = 0$. Moreover, we have found that the flat solution ($\alpha = 0$) has energy $E \approx 3.861$. Let us mention some properties of these solutions.

For $\alpha > 0$ their pattern is similar to the one occuring in case I. The critical value of $\alpha$ is larger than in I with $\alpha_{cr} \approx 0.1834$. (The solutions of case I are heavier and so exist for a smaller interval of the gravitational coupling).

Note that

$$
\begin{aligned}
E(\alpha = \alpha_{cr}) &\approx 3.377, \\
A_0(\alpha = \alpha_{cr}) &\approx 0.517, \\
C_m(\alpha = \alpha_{cr}) &\approx 0.614.
\end{aligned}
$$

We also have two branches and as $\alpha \to 0$ the upper branch solution again converges to the SU(3) Einstein sphaleron solution.

Some technical problems are encountered as $F_0'(r)$ vanishes at $r = 0$. Thus the "shooting" parameter is then $F_0''(r)|_{r=0}$.

We have also found that $F_0''(r)|_{r=0}$ for the solution on the upper branch is several orders of magnitude larger than its counterpart on the lower branch. Moreover, the shooting parameter $F_0''(r)|_{r=0}$ varies strongly with $\alpha$ and increases considerably when $\alpha$ increases (resp. decreases) on the lower (resp. upper) branch.

When plotting the energy density (see Fig. 1), it is clearly seen that, for a solution on the second branch, the energy density is much more peaked close to the origin in comparison with the energy density of the configuration on the first branch.

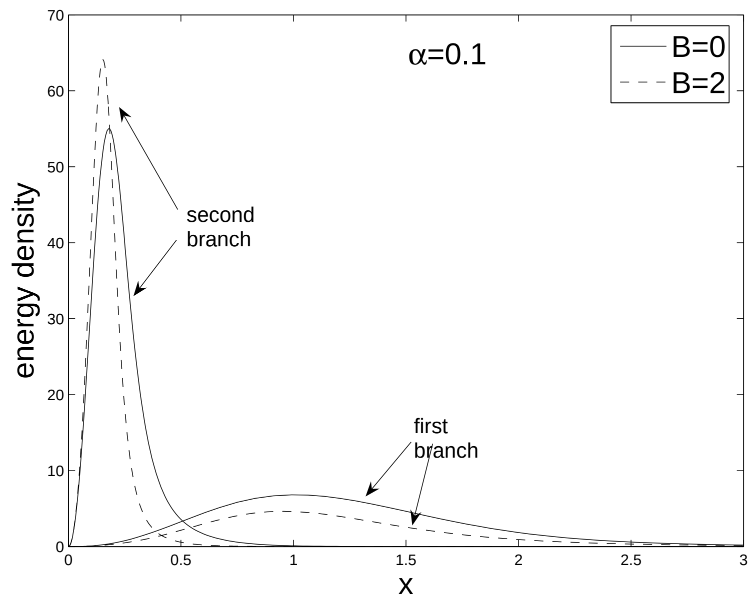


Figure 1: The energy density (in units of $\alpha^2$) of the $B = 0$ and $B = 2$ solutions (cases II and III, respectively) is shown for the configurations on the first and second branches for $\alpha = 0.1$.

## 5.4 Case III

Now $F_0(0) = \pi$, $F_1(0) = 0$ and $F_1(r)$ goes from zero to zero developping one node at some finite value of $r$.

Note that now the solutions have the topological charge 2 and have the following properties: they are the gravitating $SO(3)$ embedded solutions studied before. The regularity of the solutions at the origin implies that

$F_0'(0) = F_1'(0)$ which we have confirmed numerically. The energy of the flat solution is $E \approx 2.376$. The gravitational case again has two branches of solutions merging at a critical value of $\alpha_{cr} \approx 0.2333$.

The gravitating solution can be characterized by

$$\begin{aligned} &E(\alpha = \alpha_{cr}) \approx 2.033, \\ &A_0(\alpha = \alpha_{cr}) \approx 0.460, \\ &C_m(\alpha = \alpha_{cr}) \approx 0.537. \end{aligned} \tag{32}$$

Again, the energy density of the second branch solution is stronger peaked close to the origin in comparison with the first branch solution (see Fig. 1). Moreover, while the maximum of the energy density for the $B = 2$ solution on the first branch is smaller than that of the $B = 0$ solution on the first branch, this is the other way around on the second branch. Note, however, that the energy density of the $B = 2$ solution on the second branch is less broad than that of the $B = 0$ solution.

# 6 Conclusion

Summarising our results, we have found that the Harmonic Map method works well and that one can find exact solutions of the relevant equations. Our numerical results show that the solutions (for the matter fields) are not very different from those of the flat case. and that the higher the energy of the solution, in the flat limit, the smaller is the interval of $\alpha$ for which the gravitating skyrmions exist. Furthermore, we also have a second branch of solutions with interesting properties. On this branch the metric functions do not develop a zero in the critical limit and do not represent solutions with horizons. The matter fields, however, become singular at the origin suggesting (after a change of scale) that they can describe “gravitating sphalerons”. We remark that a similar discussion can be given for the Einstein-Yang Mills-Higgs system

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