Spectral Properties and Irreducibility of the Field Operators Set in Noncommutative Quantum Field Theory

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Abstract

It has been proved that in noncommutative theory the cyclicity of the vacuum state leads to irreducibility of the field operators set as well as in commutative case. The validity of Reeh-Schlieder theorem has been demonstrated.

1 Introduction

The axiomatic approach to quantum field theory (QFT) built up by Wightman, Jost, Bogoliubov, Haag and others made QFT a consistent, rigorous theory (for references, see [1]-[4]). In the framework of this approach, fundamental results, as the CPT and spin-statistics theorems were proven. In addition, the axiomatic formulation of QFT has given the possibility to derive analytical properties of scattering amplitudes and, as a result, dispersion relations. Consequently, various rigorous bounds on the high-energy behaviour of scattering amplitudes were obtained.

At present, noncommutative quantum field theory (NC QFT) attracts a great deal of attention. The study of such theories has got a considerable impetus after it was shown that they appear naturally, in some cases, as low-energy effective limits of open string theory in the presence of a constant antisymmetric background field [5]. In this context, the coordinate operators of a noncommutative space-time satisfy the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix of dimension $(\text{length})^2$.

The implications of the modern ideas of noncommutative geometry [6] in physics have been lately of great interest, though attempts can be traced back as far as 1947 [7]. Plausible new arguments for studying NC QFT have been offered in [5, 8] (for a review, see [9]).

We first shall consider the case of space-space noncommutativity, i.e. $\theta_{0i} = 0$, since theories with space-time noncommutativity can be obtained as low-energy effective limits from string theory only in special cases [10]. Besides, there are problems with unitarity [11] and causality [12, 13] in the general case.

Up to the present time, the study of NC QFT has been mostly done in the Lagrangian approach (for a review, see [9]). However, it is of importance to develop also an axiomatic formulation of NC QFT, which does not refer to a specific Lagrangian.

The first step in this direction was made in [14]. Wightman functions approach was developed in NC QFT in [15] - [17]. In [14] - [17] the case of space-space NC theory was considered.

As our formulation is based on the description of NC QFT in terms of Wightman functions let us recall the essence of this construction in commutative case. For simplicity we consider the case of scalar Hermitian field, a complex field can be treated by the similar way.

Vacuum vector $|0\rangle \equiv \Psi_0$ is the cyclic vector for polynomial algebra of interacting fields operators $\varphi\,(x)$. That is every vector, belonging to the space in question H, can be approximated by the vectors of a type

$$\Psi = \varphi(x_1)\,\varphi(x_2)...\varphi(x_n)\,\Psi_0 \tag{2}$$

with an arbitrary accuracy. For two arbitrary basic vectors

$$\Xi = \varphi(x_j)...\varphi(x_1)\,\Psi_0 \quad \text{and} \quad \Psi = \varphi(x_{j+1})...\varphi(x_n)\,\Psi_0$$

we have

$$\langle \Xi, \Psi \rangle = \langle \Psi_0, \varphi(x_1)...\varphi(x_n)\, \Psi_0 \rangle \equiv W\,(x_1, x_2, ..., x_n). \tag{3}$$

It is evident that every scalar product in H is approximated by linear combinations of Wightman functions with an arbitrary accuracy.

Let us proceed to noncommutative field theory.

In the case of $\theta_{0i} = 0$, choosing the coordinate system so that $\theta_{3i} = 0$, only the component $\theta_{12} = -\theta_{21} = \theta$ is nonzero. Thus in this case we have two commutative variables x_0 and x_3 and two noncommutative ones - $\hat{x}_1$ and $\hat{x}_2$. As before we admit that Ψ_0 is a cyclic vector, but now we consider the vectors of the type

$$\Psi = \varphi(\hat{x}_1)\, \varphi(\hat{x}_2)...\varphi(\hat{x}_n)|0\rangle. \tag{4}$$

Only for simplicity we treat $\varphi\,(\hat{x})$ in such a way if it were a local operator. As in commutative case we can make our consideration rigorous by substitution: $\varphi\,(\hat{x}) \to \varphi_f \equiv \int \varphi\,(\hat{x})\, f(\hat{x})\, d\,\hat{x}$, $f(\hat{x})$ is a test function.

Let us recall that in commutative case we have the following local commutativity condition:

$$[\varphi\,(x), \varphi\,(y)] = 0, \quad \text{if} \quad (x-y)^2 < 0. \tag{5}$$

The existence of two commutative coordinates x_0 and x_3 gives the possibility to substitute condition (5) by the following one [19]:

$$[\varphi\,(\hat{x}), \varphi\,(\hat{y})] = 0, \quad \text{if} \quad (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \tag{6}$$

It is well-known that condition (5) plays the crucial role in proving of such results as CPT-theorem, spin-statistic theorem [1] - [3] as well as analytical properties of scattering amplitude. In noncommutative theory condition (6) (or its generalization) plays the same role [14] -[17], [20] - [24]. However, some important general results in usual QFT were obtained without use of condition (5), they are the consequences of spectral properties of Wightman functions only. On this ground the irreducibility of the set of field operators has been proved [1, 2]. These spectral properties lead to the analyticity of Wightman functions in tubes [1] - [3], which in its turn is the basis of the derivation of Reeh-Schlieder theorem. In accordance with this theorem the set of Wightman functions is "almost defined" by the corresponding set of Wightman function, in which all x_i belong to arbitrary open domain O.

In this report we concentrate our efforts on results in noncommutative theory based on spectral properties of Wightman function. We show that as well as in commutative case the set of field operators is irreducible one if the vacuum vector is a cyclic one. We also obtain the noncommutative analog of Reeh-Schlieder theorem [18].

2 Irreducibility of the Field Operators Set

First let us recall that in commutative case spectral property can be expressed as follows:

$$W\,(P_1, ...P_{n-1}) = \\ = \frac{1}{(2\pi)^{2\,(n-1)}}\int e^{i\,P_j\,\xi_j}\,W\,(\xi_1, ...\xi_{n-1})\,d\,\xi_1...d\,\xi_{n-1} \neq 0 \quad (7)$$

only if $P_i \in \bar{V}^+\ \forall\ i$, i.e. if $P_i^0 \geq |\vec{P}_i|$. This condition is the consequence of the fact that there are no tachyons in the set of basic vectors. This condition leads immediately to the analyticity of Wightman functions in the tube T_n^- that is in the domain $\nu_i = \xi_i - i\,\eta_i,\ \eta_i \in \bar{V}^+,\ \xi_i \in \mathbb{R},\ \xi_i \equiv x_i - x_{i+1}$. Owing to translation invariance $W\,(x_1, x_2, ..., x_n) = W\,(\xi_1, ..., \xi_{n-1})$.

Proceeding to noncommutative case we first consider the case of space-space noncommutativity. In this case we have the analogous condition with the only difference that integration has to be taken only over commutative coordinates ξ_i^0 and ξ_i^3. The condition for P_i is changed respectively: the integral in (7) is not zero only if $P_i \in \bar{V}_2^+$, that is if $P_i^0 > |P_i^3|$ as we assume that all basic vectors belong to $\bar{V}_2^+$.

This condition leads immediately to the analyticity of noncommutative Wightman functions in the tube T_n^-. Here $T_n^-$is the tube in respect to commutative coordinates.

In commutative case spectral property is the consequence of the condition [1, Ch. 2]

$$\int d\,a\,e^{i\,p\,a}\,\langle\Phi, U\,(a)\,\Psi\rangle \neq 0, \quad \text{only if} \quad p \in \bar{V}^+, \qquad (8)$$

Φ and Ψ are arbitrary vectors, $U\,(a)$ is a translation operator.

In noncommutative case we have the same condition, but now $p \in \bar{V}_2^+,\ a$ is a translation of commutative variables.

Let us remind the criterion of irreducibility. As $\varphi(\hat{x})$is an unbounded operator, a proper definition of irreducibility would be:

Definition. The set of operators $\varphi(\hat{x}_i)$ is irreducible if any bounded operator A, which commutes with all field operators

$$[A, \varphi(\hat{x})] = 0, \qquad \forall\, \hat{x} \tag{9}$$

is the following:

$$A = C\mathbf{I}, \qquad C \in \mathbb{C}, \tag{10}$$

$\mathbf{I}$ is an identical operator.

To prove the irreducibility of the set of operators in question we consider the expression

$$\langle \Psi_0, A\, U(a)\, \varphi(\hat{x}_1)\, \varphi(\hat{x}_2) ... \varphi(\hat{x}_n)\, \Psi_0 \rangle.$$

Using translation invariance of Ψ_0 and condition (9), we can write the chain of equalities:

$$\begin{aligned} &\langle \Psi_0, A\, U(a)\, \varphi(\hat{x}_1)\, \varphi(\hat{x}_2) ... \varphi(\hat{x}_n)\, \Psi_0 \rangle = \\ &\qquad = \langle \Psi_0, \varphi(\hat{x}_1 + a) \cdots \varphi(\hat{x}_n + a)\, A\, \Psi_0 \rangle = \\ &\qquad\qquad = \langle \varphi(\hat{x}_n) \cdots \varphi(\hat{x}_1), U(-a)\, A\, \Psi_0 \rangle. \end{aligned} \tag{11}$$

Let us remind that $\varphi(\hat{x}_i)$ is a Hermitian operator, a is a translation in respect to commutative variables. So

$$\langle A^+ \Psi_0, U(a)\, \varphi(\hat{x}_1) ... \varphi(\hat{x}_n)\, \Psi_0 \rangle = \langle \varphi(\hat{x}_n) \cdots \varphi(\hat{x}_1)\, \Psi_0, U(-a)\, A\, \Psi_0 \rangle. \tag{12}$$

Let us consider

$$\begin{aligned} &\int d\, a\, e^{i p a}\, \langle A^+ \Psi_0, U(a)\, \varphi(\hat{x}_1) ... \varphi(\hat{x}_n)\, \Psi_0 \rangle = \\ &\qquad \int d\, a\, e^{i p a}\, \langle \varphi(\hat{x}_n) \cdots \varphi(\hat{x}_1), U(-a)\, A\, \Psi_0 \rangle. \end{aligned} \tag{13}$$

In accordance with eq. (8) the left part of eq. (13) is not zero only if $p \in \bar{V}_2^+$ and the right part is not zero only if $-p \in \bar{V}_2^+$. So if we take vector $\varphi(\hat{x}_n) \cdots \varphi(\hat{x}_1)\, \Psi_0$, which is the linear combination of all basic vectors excluding vacuum one, then

$$\int d\, a\, e^{i p a}\, \langle \varphi(\hat{x}_n) \cdots \varphi(\hat{x}_1), U(-a)\, A\, \Psi_0 \rangle = 0 \quad \forall\, p. \tag{14}$$

Eq. (14) implies that

$$\langle \Phi, A\,\Psi_0 \rangle = 0 \tag{15}$$

for any vector Φ, which is the linear combination of all basic vectors excluding vacuum one.

So

$$A\,\Psi_0 = C\,\Psi_0, \qquad C \in \mathbb{C}. \tag{16}$$

From eq. (16) it follows immediately that

$$A\,\varphi(\hat{x}_1)\,...\varphi(\hat{x}_n)\,\Psi_0 = C\,\varphi(\hat{x}_1)\,...\varphi(\hat{x}_n)\,\Psi_0. \tag{17}$$

As A is a bounded operator eq. (17) implies that

$$A\,\Phi = C\,\Phi, \qquad \forall \Phi \in H. \tag{18}$$

Let us proceed to the general case $\theta_{0i} \neq 0$. As translation invariance is valid in noncommutative theory, we can obtain the same result for general case as well. To do it we use the possibility to describe noncommutative fields in commutative coordinate space. To this end we have to substitute the usual operators product by $\star$ (Moyal) product [9]. The extension of this $\star$ product to different points $\hat{x}_1$ and $\hat{x}_2$ is not unique. In [14] it was assumed that in different points we can use a standard multiplication. In [15] $\star$ product was proposed also for different points. In [17] it was shown that a concrete form of above mentioned multiplication is not essential if it satisfies some general constrain. Such multiplication we denote as $\varphi\,(x)\tilde{\star}\varphi\,(y)$. So

$$W\,(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) = \langle \Psi_0, \varphi\,(x_1)\,\tilde{\star}\,\varphi\,(x_2) \cdots \tilde{\star}\,\varphi\,(x_n)\,\Psi_0 \rangle. \tag{19}$$

If we assume that basic vectors are the eigenvectors of the translation operator in the usual meaning, that is

$$U\,(a)\,\Psi_P = e^{i\,P\,a}\,\Psi_P, \tag{20}$$

we see immediately that standard proof of eq. (8) [1, Ch. 2] is valid in noncommutative case as well.

We have shown above that condition $P_n^0 > 0$ for any basic vector, excluding vacuum one, is sufficient to prove irreducibility in noncommutative case. Thus if this condition is valid in general noncommutative case, then corresponding set of field operators is irreducible.

3 Reeh-Schlieder Theorem in Noncommutative Theory

Let us first consider the case of space-space noncommutativity. Let O be an arbitrary open domain of commutative variables x_i^0 and x_i^3. We prove that if

$$\langle \Phi, \varphi(\hat{x}_1) \dots \varphi(\hat{x}_n)\, \Psi_0 \rangle = 0 \quad \forall (x_i^0, x_i^3) \in O, \tag{21}$$

then $\Phi = 0$.

For simplicity we restrict ourselves by vectors $\Phi \in H_0$, where H_0 is the space of all finite linear combinations of basic vectors, H is a closure of H_0. In accordance with above mentioned analytical properties of noncommutative Wightman functions in tubes,

$$\langle \Phi, \varphi(\hat{x}_1) \dots \varphi(\hat{x}_n)\, \Psi_0 \rangle$$

is an analytical function of commutative variables $-x_1 - i\,\eta_1,\ x_1 - x_2 - i\,\eta_2 \cdots x_{n-1} - x_n - i\,\eta_n,\ \eta_i \in \bar{V}_2^+,\ x_i = \{x_i^0, x_i^3\}$. Owing to the condition (21) this function vanishes in the open set on its boundary. Thus this function is equal to zero in the all domain of analyticity and all boundary points.

So

$$\langle \Phi, \varphi(\hat{x}_1) \dots \varphi(\hat{x}_n)\, \Psi_0 \rangle = 0 \quad \forall \hat{x}_i. \tag{22}$$

As vectors $\varphi(\hat{x}_1) \dots \varphi(\hat{x}_n)\, \Psi_0\rangle$ form the whole set of basic vectors, the condition (22) means that Φ.

Now let us proceed to the general case. If we assume that $P_n^0 > |P_n^j|$, where j is one of spatial coordinates, Wightman function is an analytical function in the tube: $\nu_i = \xi_i - i\,\eta_i,\ \xi_i = \{\xi_i^0, \xi_i^3\},\ \eta_i = \{\eta_i^0, \eta_i^3\},\ \eta_i \in \bar{V}_2^+$. So we can obtain Reeh-Schlieder theorem, in which O is the open domain in variables x_i^0, x_i^j. Let us point out that in commutative case our result means that Reeh-Schlieder theorem is still valid if we use a smaller domain O than standard one.

It follows from Reeh-Schlieder theorem that the set of operators $\varphi\,(x_i)$, $x_i \in O$ and Π, where Π is a projection operator on the vacuum state, is irreducible. The proof is similar with the proof of analogous assertion in commutative case [1].

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