Gravity in the Braneworld and the AdS/CFT Correspondence

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Abstract

We discuss gravitational interaction realized on the Randall-Sundrum (RS) infinite braneworld. The RS infinite braneworld has an outstanding feature that effects of extra-dimension are not suppressed exponentially at a long distance. Another interesting aspect is that the model allows a dual interpretation according to the AdS/CFT correspondence conjecture. We show several explicit confirmations of the AdS/CFT correspondence on the correction to the gravitational interaction, including a new result about tensor perturbations on a Friedmann brane. Based on the AdS/CFT correspondence the author has pointed out the possibility that black holes evaporate in a classical manner. We also mention the current status of study related to this conjecture.

1 Introduction

The braneworld scenario was proposed as an alternative way of compactifying extra dimensions[1, 2, 3, 4]. The braneworld scenario is essentially different from the ordinary Kaluza-Klein compactification in that the matter fields of the standard model are localized on the brane, while gravity can propagate in a higher dimensional spacetime called "bulk". Since the ordinary matter fields are localized on the brane, they do not notice the presence of extra-dimensions. Particle physics experiments are not affected by extra-dimensions before the energy reaches the scale determined by the size of the extra-dimensions. Hence we do not see any contradiction even though extra dimensions are rather large. In such a scenario, gravity can be modified at a relatively long distance. The experimental constraint on deviation from Newton's law is absent below sub *mm* scale. Naively therefore there is a possibility that extra dimensions are as large as sub *mm* scale.

A novel idea was proposed by Randall and Sundrum (RS)[3, 4], and we focus on their second model (RS II)[4]. The model assumes 5D Einstein gravity with a negative cosmological constant $\Lambda = -6/\ell^2$. Ordinary matter fields live on a 4-dimensional brane, which has positive tension $\sigma = 3/4\pi G_5 \ell$, where G_5 is the 5D Newton constant. Z_2-symmetry is imposed across the brane.

The simplest background solution is 5D anti-de Sitter (AdS) space

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \tag{1}$$

with a brane placed at $z = \ell$. Here $\eta_{\mu\nu}$ is the 4D Minkowski metric. An outstanding feature of this model is that 4D Einstein gravity is effectively reproduced on the brane in spite of the infinite extension of the extra dimension.

2 Linear analysis of Randall-Sundrum II model

First we briefly discuss linear perturbations of this model. For this purpose, the RS gauge ($h_{za} = 0$ with transverse-traceless conditions) is convenient. In this gauge, the equations for bulk metric perturbations become

$$\left[-\partial_z^2 + V(z)\right]\psi_{\mu\nu} = \Box\psi_{\mu\nu}, \tag{2}$$

where $\psi_{\mu\nu} = \sqrt{|z| + \ell}\, h_{\mu\nu}$ and

$$V(z) = \frac{15}{4(|z|+\ell)^2} - 3\ell^{-1}\delta(z). \tag{3}$$

Assuming the form $\psi_{\mu\nu} \propto u_m(z)e^{ik_\mu x^\mu}$, we have an eigen value problem to determine the mode $u_m(z)$. Here $m^2 = -k_\mu k^\mu$ is interpreted as the mass of an effective 4D field. An interesting fact is that the 5D metric given in

Eq. (1) satisfies the 5D Einstein equations even if we replace the Minkowski metric $\eta_{\mu\nu}dx^\mu dx^\nu$ with any vacuum solution of the 4D Einstein equations. Correspondingly, a discrete massless mode $u_0(z)$ exists. The massless wave function behaves as $u_0(z) \propto 1/z^2$. Since the potential $V(z)$ vanishes at $|z| \to \infty$, the mass spectrum is continuous for $m^2 > 0$. The potential $V(z)$ has a barrier near the brane with the height of $O(\ell^{-2})$. The wave function $u_m(z)$ with $0 < m \lesssim \ell^{-1}$ is suppressed near the brane due to this potential barrier. On the other hand, $u_m(z)$ with $m \gtrsim \ell^{-1}$ is unsuppressed near the brane, but excitation of such modes requires a large input of energy. Therefore the zero mode dominates at low energies, and hence 4D Einstein gravity is recovered at the linear level.

Linear metric perturbations induced on the brane were fully explicitly shown in Ref.[5] as

$$h_{\mu\nu} = -16\pi G_5 \int d^4x' G(x,x') \left(T_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu} T \right) + \frac{8\pi G_5 \ell^{-1}}{3} \gamma_{\mu\nu} {}^{(4)}\Box^{-1} T, \quad (4)$$

where $G(x,x')$ is the 5D scalar Green function.

In the case of spherical symmetry the gravitational field outside the matter distribution is evaluated as $h_{00} \approx \frac{2G_4M}{r}\left(1 + \frac{2\ell^2}{3r^2}\right)$, $h_{ij} \approx \frac{2G_4M}{r}\left(1 + \frac{\ell^2}{3r^2}\right)$ [5, 6]. The correction to 4D Einstein gravity is suppressed by ℓ^2/r^2, where r is the distance from the source. If we neglect the contribution due to massive modes ($m^2 > 0$), Eq. (4) exactly recovers linearized 4D Einstein gravity.

3 Non-Linear Perturbation

Recovery of 4D Einstein gravity at the non-linear level is not so trivial. The difficulty can be anticipated from the linear analysis. The wave function of the massless mode behaves as

$$h_{\mu\nu}(\text{massless mode}) \approx \frac{1}{z^2}, \quad (5)$$

at a large z, and hence a Weyl invariant $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ behaves as $\approx z^4$. In the direction of the extra dimension therefore the curvature becomes infinitely large. The wave function of the massive modes behaves as

$$h_{\mu\nu}(\text{massive mode}) \approx \frac{1}{\sqrt{z}}. \tag{6}$$

Hence, the same invariant is more severely divergent like $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \approx z^7$.

However, such divergences do not directly indicate breakdown of the perturbation analysis. For static cases, the perturbed metric induced by matter fields on the brane was shown to be regular at large z[5] . Even in dynamical cases, the asymptotic regularity was proved in Ref.[7]. Here it will be worth pointing out that the regularity is recovered only after summation over all massless and massive modes.

Analysis of non-linear perturbations is not straightforward. The picture of a 4D effective theory composed of a tower of massive gravitons breaks down. In this picture the effective coupling between various massive gravitons diverges. Nevertheless, for static and spherically symmetric configurations on the brane second order perturbations were proved to behave well[8, 9]. Approximate recovery of 4D Einstein gravity in the strong gravity regime is also confirmed numerically in Ref.[10].

We found that 4D Einstein gravity is a rather good approximation to gravity on the brane in the R-S II model. However, there still remains a question whether the similarity to 4D Einstein gravity continues to hold even when an event horizon is formed.

4 Black Hole and AdS/CFT Correspondence

In Ref.[11], a black string solution given by

$$ds^2 = \frac{\ell^2}{(|z| + \ell)^2}\left[dz^2 + q^{(4)}_{\mu\nu}dx^\mu dx^\nu\right], \tag{7}$$

was discussed, where $q^{(4)}_{\mu\nu}dx^\mu dx^\nu$ is the usual 4D Schwarzshild metric. For this solution $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ behaves like $\propto z^4 r^{-6}$, where r is the Schwarzshild radial coordinate. At a large z for a fixed r, this Weyl invariant diverges[11]. Also, this configuration is known to be unstable[12, 13]. Hence, the black string solution cannot be the final state of gravitational collapse. Therefore it is natural to expect that another black hole solution whose event horizon is localized near the brane will exist.

However, this expectation might be too naive. I raised a conjecture that black holes localized on the brane may not exist[14], based on the AdS/CFT correspondence conjecture[15, 16]. The AdS/CFT correspondence conjecture indicates

$$W_{CFT} = S_{EH} + S_{GH} - S_1 - S_2 - S_3, \tag{8}$$

where $S_{EH} = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left({}^{(5)}R + \frac{12}{\ell^2}\right)$,
$S_{GH} = -\frac{1}{8\pi G_5}\int d^4x\sqrt{-{}^{(4)}g}K$, $S_1 = -\frac{3}{8\pi G_5 \ell}\int d^4x\sqrt{-{}^{(4)}g}$, $S_2 = -\frac{\ell}{32\pi G_5}\int d^4x\sqrt{-{}^{(4)}g}\,{}^{(4)}R$ and $S_3 = \cdots$. W_{CFT} is the effective action of 4D CFT fields evaluated on the metric induced on the boundary. K is the trace of the extrinsic curvature of the boundary. Left hand side is the action for 5D gravity theory. The counter terms S_1, S_2 and S_3 are necessary to cancel manifest dependence on the boundary location. The location of the boundary plays the role of the cutoff parameter of CFT. Now we consider the action of the RS II model without any matter fields:

$$S_{RS} = 2\left(S_{EH} + S_{GH}\right) - 2S_1 = 2S_2 + 2\left(W_{CFT} + S_3\right). \tag{9}$$

Here we used the formula (8) in the second equality. We notice that $2S_2$ is nothing but the ordinary 4D Einstein-Hilbert action, while $W_{CFT} + S_3$ is the effective action of cutoff CFT. This formula indicates equivalence between the RS infinite braneworld and 4D Einstein gravity with cutoff CFT. Here we call this statement an extended AdS/CFT conjecture. Notice that the one-loop quantum effect of 4D CFT fields is included in the classical dynamics of bulk gravity in 5D picture.

The correspondence at the linear level was clearly shown in Ref.[17]. After integrating out the CFT degrees of freedom, we obtain the effective action for 4D gravity as

$$S \approx \frac{1}{2}\int d^4x\, \psi^{\mu\nu}\left(D^{-1}_{\mu\nu\alpha\beta} + \Pi_{\mu\nu\alpha\beta}\right)\psi^{\alpha\beta},$$

where D^{-1} is the graviton propagator and Π is the CFT contribution to the graviton self-energy. From this action we obtain an equation schematically like $\psi \approx DT - D\Pi DT$, where T is the matter energy-momentum tensor. In this manner the leading correction to 4D Einstein gravity in momentum representation is evaluated as

$$\delta\tilde{h}_{\mu\nu} = -16\pi G\Pi(p)\left\{\tilde{T}_{\mu\nu}(p) - \frac{1}{3}\eta_{\mu\nu}\tilde{T}(p)\right\},$$

where $\Pi(p) = -\frac{\ell^2}{4}(\ln(p^2/\mu^2) + constant)$. A quantity associated with "$\sim$" is the Fourier transform of the corresponding variable.

On the other hand, in the 5D picture the leading correction is given by

$$\delta\tilde{h}_{\mu\nu} = 8\pi G_5 \frac{K_0(p\ell)}{pK_1(p\ell)} \left\{ \tilde{T}_{\mu\nu}(p) - \frac{1}{3}\eta_{\mu\nu}\tilde{T}(p) \right\}.$$

By expanding the modified Bessel functions as $K_0(p\ell)/pK_1(p\ell) \approx -\frac{\ell}{2}(\ln(p^2\ell^2/4) + constant) + O(p^2\ell^4)$, we find that the correspondence holds perfectly at the linear level. The difference starts with the forth power in ℓ due to higher order corrections that we neglected here.

In general the correspondence has not been verified beyond the linear level. However, homogeneous cosmology gives an important exceptional example[18]. By using the geometric approach[19], the trace of the effective Einstein equations is written without any reference to the bulk as ${}^{(4)}G = 8\pi G_4 T + \frac{(8\pi G_4 \ell)^2}{4}\left(T_{\mu\nu}T^{\mu\nu} - \frac{1}{3}T^2\right)$. On the other hand, the trace part of the energy momentum tensor of CFT is solely determined by the trace anomaly, and hence the trace of the Einstein equation with CFT becomes ${}^{(4)}G = 8\pi G_4 T + \ell^2/4\left({}^{(4)}G_{\mu\nu}{}^{(4)}G^{\mu\nu} - \frac{1}{3}{}^{(4)}G\,{}^{(4)}G\right)$. Hence, it will be easy to see that the difference starts with the forth power in ℓ. Therefore approximately the same modified Friedmann equation follows in both cases.

Moreover, we have recently shown that the correspondence holds not only for the background Friedmann cosmology but also for perturbations on it[20]. In both pictures the same equation governing tensor type perturbations with a comoving wavenumber k^μ,

$$\begin{aligned} \left(\partial_\eta^2 + 2\mathcal{H}\partial\eta + k^2\right) h_{ij} &= \frac{\ell^2}{a^2}\left[\left(3\mathcal{H}^3 - 2\mathcal{H}\mathcal{H}'\right)\partial_\eta + k^2\mathcal{H}^2\right] h_{ij} \\ &\quad -\frac{\ell^2}{2a^2}\int d^4p\,\tilde{h}_{ij}e^{-i\omega\eta}p^4\ln\left(\frac{p}{a\mu}\right), \end{aligned}$$

is derived. Here η is the conformal time, $\mathcal{H} = \partial_\eta \ln a$, $p^2 = -\omega^2 + k^2$.

So far, we have observed several direct evidences for the extended AdS/CFT correspondence, which we now apply to the formation of a black hole in the RS II model. In 4D CFT picture, the back reaction due to the Hawking radiation is expected to be much more efficient than in the ordinary 4D theory by a factor of ℓ^2/G_4. According to the AdS/CFT correspondence, the one-loop quantum effect of CFT in 4D picture must be described as a classical dynamics in 5D picture. This indicates classical black hole evaporation and the absence of a stationary black hole solution in the 5D RS model[14, 21].

Then, the first question is whether a brane black hole looks similar to an ordinary 4D BH or not. It is very interesting if a brane black hole looks quite different from the ordinary black hole in the 4D Einstein gravity. In this case we might be able to use astronomical observations of black holes to probe the extra dimension. If they look similar, there arises the second question whether the AdS/CFT correspondence applies for the black hole configuration. Even the case that the correspondence does not hold in such a strong gravity regime is still interesting because we may be able to distinguish the model described by the 4D CFT picture from the 5D RS model observationally in this case. Extremely interesting is the remaining possibility that the correspondence really applies even for a black hole configuration. In this case, we can evaluate the mass loss rate due to the Hawking radiation as $\dot{M}/M \sim N \times (1/G_4^2 M^3) \sim \ell^2/(G_4 M)^3$. Then the evaporation time scale becomes $\tau = (M/M_\odot)^3 (1mm/\ell)^2 \times 1$ year. The existing black holes in X-ray binaries will give a stronger constraint on the value of ℓ than that from the laboratory experiment[22].

There is a static black hole solution when a 2-brane in the 4 dimensional bulk is considered[23]. The 3-dimensional metric induced on the brane looks similar to a 4-dimensional Schwarzshild black hole;

$$ds^2 = -\left(1 - \frac{2\mu\ell}{r}\right) dt^2 + \left(1 - \frac{2\mu\ell}{r}\right)^{-1} dr^2 + r^2 d\varphi^2. \tag{10}$$

However, the period in φ-direction is not 2π but $\Delta\varphi \approx \dfrac{4\pi}{3(2\mu)^{1/3}}$, where we assumed that $\mu \gg 1$.

One may think that the presence of this static black hole solution in a lower dimensional case is a counter example against the conjecture of the classical black hole evaporation. But this does not apply at all. An important peculiarity of this lower dimensional example is that the induced metric is far different from a solution of the 3D vacuum Einstein equations. It is worth pointing out that this effective energy momentum tensor necessary to realize the above solution

$$T_{\mu\nu}^{CFT} = -\frac{\mu\ell}{8\pi G_3 r^3} \,\mathrm{diag}(1, 1, -2), \tag{11}$$

can be understood as the Casimir energy of CFT on a background of 3D flat spacetime with a deficit angle, which is in fact a solution of the 3D vacuum

Einstein equations. This means that the black hole horizon is formed only after taking into account the one-loop effect due to CFT fields. Namely, at the lowest order there is no black hole. Therefore it is natural that the CFT fields do not have any radiating component corresponding to the Hawking radiation[21].

Finally, we give a few comments on recent attempts to find a black hole solution by means of a numerical method[24, 25]. Small black hole solutions whose horizon radii are smaller than or comparative to the AdS curvature radius ℓ were found, and an interesting indication against the classical black hole evaporation conjecture was obtained. In the above Refs. a few thermodynamic relations, like a temperature-area relation, were plotted. Those plots indicate that the sequence of numerical solutions can be smoothly extrapolated to those who look like a 4D Schwarzshild black hole. However, it became more and more difficult to construct larger black hole solutions.

Thus, first of all, a question to answer is whether the small black hole solutions obtained numerically really exist in a strict sense. As we have seen before, the difference between two different 4D and 5D pictures arises at the forth power in ℓ. Hence, when the size of black hole is as small as ℓ, the naive correspondence will cease to hold. Therefore the presence of small black hole solutions does not contradict with the classical black hole evaporation conjecture. There is a possibility that the answer to the above question is "No". Even if tiny inconsistencies may exist, they can be hidden by numerical errors. If the answer is "Yes", there arises the second question whether large black holes continue to exist or not. To answer the second question, further study is necessary.

5 Summary

First of all gravity in the RS II model mimics 4D Einstein gravity very well. Although a naive picture to see this model as the usual four dimensional Einstein gravity with a tower of massive gravitons breaks down in non-linear regime, the effets of the fifth dimension still remain small. However, interestingly the extended AdS/CFT correspondence suggests a qualitative difference from the standard case in that there is no static large brane black hole solution. Alghough the extended AdS/CFT correspondence was explicitly checked in various cases, such as liner perturbations on a Minkowski brane, the modified Friedmann equation, and tensor perturbations on a Friedmann brane, there still remains the possibility that the naive correspondence does not work once a black horizon is formed. To examine whether black hole solution exists or not, numerical calculation was performed. We succeeded in constructing small brane black holes, whose horizon size is not much bigger than the bulk curvature scale. In this regime, the AdS/CFT correspondence is expected to be invalidated. Hence, further study is necessary to settle this issue.

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