Younger sisters of the Seiberg–Witten effective theory

A. V. Smilga

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SUBATECH, Université de Nantes,
4 rue Alfred Kastler, BP 20722, Nantes 44307, France. [1]

Abstract

We consider the theories obtained by dimensional reduction to $D = 1, 2, 3$ of $4D$ supersymmetric Yang–Mills theories and calculate there the effective low-energy lagrangians describing moduli space dynamics — the low-dimensional analogs of the Seiberg–Witten effective lagrangian.

1 Introduction

Effective lagrangians arise naturally in theories involving two energy scales. Integrating out the "fast" variables (the degrees of freedom with large characteristic excitation energy), one obtains the effective lagrangian involving only "slow" variables, which describes the low–energy dynamics. This note represents a short version of the review [1] devoted to the effective lagrangians of supersymmetric gauge theories in the dimensions 3,2, and 1. They have many things in common with the effective supersymmetric lagrangians in four dimensions and, in particular, with the Seiberg-Witten effective lagrangian [2].

Let us start with reminding of its salient features. Consider the pure $4D$ $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. The lagrangian written in terms

[1]On leave of absence from ITEP, Moscow, Russia.

of $\mathcal{N}=1$ superfields is

$$\mathcal{L} \;=\; \frac{1}{g^2}\mathrm{Tr}\left\{\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar\theta\, \Phi e^{-V}\bar\Phi e^{V}\right\} \qquad (1)$$

In the bosonic sector, it includes the gauge field A_μ and a complex scalar ϕ belonging to the adjoint representation of the gauge group,

$$g^2\mathcal{L} \;=\; -\frac{1}{2}\mathrm{Tr}\{F_{\mu\nu}^2\} + 2\mathrm{Tr}\{\mathcal{D}_\mu\bar\phi\mathcal{D}_\mu\phi\} - \mathrm{Tr}\{[\bar\phi,\phi]^2\} \;+\text{fermions} \qquad (2)$$

This theory has (infinitely) many different classical vacua. The supersymmetric vacuum has zero energy. At the classical level, it has zero potential energy. The scalar potential vanishes when $[\bar\phi,\phi]=0$, implying that ϕ belongs to the Cartan subalgebra of the corresponding Lie algebra. Factorizing over gauge transformations, this gives r physical complex parameters (where r is the rank of the group) characterizing the classical vacuum moduli space. When quantum corrections are taken into account, one could in principle expect the appearance of a non-trivial effective potential on the moduli space, such that the energy would generically be shifted from zero. For supersymmetric theories, quantum corrections *vanish* at any order of perturbation theory; for the $\mathcal{N}=2$ theory, non-perturbative corrections to the effective potential also vanish. However, corrections to the kinetic part of the lagrangian need not vanish and they do not. The relevant slow variables are r complex parameters ϕ^A, mentioned above and their $\mathcal{N}=2$ superpartners involving fermions, and also r Abelian gauge fields. In the simplest $SU(2)$ case, they can be combined into one $\mathcal{N}=2$ chiral superfield $\mathcal{W}=\phi+\ldots$ The effective Seiberg–Witten lagrangian has the form

$$\mathcal{L} = \int d^2\theta d^2\theta' F(\mathcal{W}) + \text{ c.c. }. \qquad (3)$$

When expressed in components, this gives a non-trivial metric on the moduli space.

Here we consider the effective lagrangians in the theories obtained by the dimensional reduction of (2) and also by the dimensional reduction of $\mathcal{N}=1$ SYM theories. Let us start by discussing the latter. In four dimensions, pure SYM theories do not possess a vacuum moduli space. The number of quantum vacua is finite, given by the adjoint Casimir operator c_V of the gauge group [3]. However, a moduli space does appear after dimensional reduction. Consider first the theory reduced to $(0+1)$ dimensions. In such a theory, new gauge invariants made of the spatial components of the gauge potential, like $\mathrm{Tr}\{A_i^2\}$, appear. By gauge rotation, A_i can be brought into the form $c_i t^3$. The three variables c_i characterize the vacuum moduli space. For an arbitrary gauge group, the moduli space is characterized by $3r$ parameters.

Consider now the reduction to $(1+1)$ dimensions. Only two components of A_i do not involve the derivative term in their gauge transformation law and we have $2r$ physical moduli space parameters. When reducing to $D=3$, only one component of the vector potential for each unit of rank is left, but there are also r Abelian gauge fields which are dual in three dimensions to scalars, $\epsilon_{ijk}F_{jk} \leftrightarrow \partial_i \Psi$. Thus, in three dimensions we have $r+r=2r$ parameters in the vacuum moduli space.

For $\mathcal{N}=2$ theories, the counting is basically the same, only we have to add $2r$ parameters associated with the scalar fields. In other words, the corresponding effective lagrangians involve $5r$ bosonic degrees of freedom in the $1D$ case and $4r$ degrees of freedom in the $2D$ and $3D$ cases.

2 $D=1:$ Symplectic Sigma Models

Consider the simplest example, namely massless $\mathcal{N}=1$ $4D$ SQED, The effective lagrangian (determined in [4]) depends on the gauge potentials $A_i(t)$ and their superpartners: the photino fields $\psi_\alpha(t)$, $\alpha=1,2$. The charged scalar and spinor fields represent fast variables that should be integrated over. Now A_i, the auxiliary field D and the spinor fields ψ_α can be combined in a single $\mathcal{N}=2$ $1D$ superfield [5]

$$\begin{aligned}\Gamma_k &= A_k + \bar\theta\sigma_k\psi + \bar\psi\sigma_k\theta + \epsilon_{kjp}\dot A_j\bar\theta\sigma_p\theta + D\bar\theta\sigma_k\theta \\ &\quad + i(\bar\theta\sigma_k\dot\psi - \dot{\bar\psi}\sigma_k\theta)\bar\theta\theta + \frac{\ddot A_k}{4}\theta^2\bar\theta^2 \ . \end{aligned} \tag{4}$$

The field (4) satisfies the constraints

$$D_{(\alpha}\Gamma_{\beta\gamma)} \ = \ 0\,, \quad \bar D_{(\alpha}\Gamma_{\beta\gamma)} = 0 \ , \tag{5}$$

where $\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha} \ = \ i(\sigma_k)_\alpha{}^\gamma\epsilon_{\beta\gamma}\Gamma_k$ and D_α, $\bar D_\alpha$ are the covariant derivatives. Actually, Γ_k are nothing but the spatial components of the former $4D$ superconnections

$$\Gamma_\mu = \frac{1}{4}(\bar\sigma_\mu)^{\dot\beta\alpha}\bar D_{\dot\beta}D_\alpha V = \ A_\mu + \dots \ , \tag{6}$$

the covariant derivatives having the form $\partial_\mu - i\Gamma_\mu$. In one-dimensional theory, Γ_k is gauge invariant,

The effective supersymmetric and gauge–invariant action is presented in the form

$$S = \int dt \int d^2\theta d^2\bar{\theta} \; F(\Gamma_k). \tag{7}$$

By construction, it enjoys $\mathcal{N} = 2$ supersymmetry. Being expressed in components, this gives a nontrivial metric $ds^2 = hd\mathbf{A}^2$, where $h(\mathbf{A}) = -\frac{1}{2}\,\partial^2 F(\mathbf{A})$, on the moduli space. This is a supersymmetric sigma model with conformally-flat $3D$ target space. However, it is not the conventional supersymmetric sigma model associated with the de Rahm complex. The latter has only one pair of complex supercharges $(Q, Q^\dagger) \equiv (d, d^\dagger)$. When the target space represents a Kähler manifold, one can define an extra pair of supercharges (three such extra pairs for hyper–Kähler manifolds), but in our case the target space is 3–dimensional and definitely not Kähler.

A reader might be somewhat confused at this point. The widely-known theorem [6] seems to assert that $\mathcal{N} = 2$ sigma models can only be defined on Kähler manifolds (and $\mathcal{N} = 4$ models only on hyper–Kähler manifolds). However, this theorem relies on two assumptions : *(i)* the theory considered should be a real field theory with at least 2 spacetime dimensions and *(ii)* the kinetic term should have the standard form $\propto g_{ab}\partial_\mu\phi^a\partial_\mu\phi^b$. For quantum mechanics, the first condition is not satisfied and there are no restrictions.

In a standard sigma model, fermions are vectors in the tangent space. In our case, they belong to the spinor representation of $SO(3) \equiv Sp(2)$. We will call this model a symplectic sigma model of the first kind (see below for the second kind).

In our case, the function $F(\Gamma_k)$ has a particular form. At the tree level, $F(\mathbf{\Gamma}) = \mathbf{\Gamma}^2/(6e^2)$ and $h = 1/e^2$. This gives the lagrangian of dimensionally-reduced photodynamics. One can evaluate the one-loop correction to the metric. The result is (the volume restrictions force us to skip all the derivations)

$$e^2 F^{\rm SQED}(\mathbf{\Gamma}) = -\frac{\mathbf{\Gamma}^2}{3} + \frac{e^2 \ln|\mathbf{\Gamma}|}{|\mathbf{\Gamma}|} + ... \;\leftrightarrow\; e^2 h(\mathbf{A}) \tag{8}$$

$$= 1 + \frac{e^2}{2|\mathbf{A}|^3} + \ldots \tag{9}$$

Let us discuss now non–Abelian theories. In the simplest case of the group $SU(2)$, the moduli space involves the variables $c_k = A_k^3$ and their superpartners, which are combined in the superfield Γ^3_k. The effective action again has the form (7), but the function $F(\mathbf{\Gamma}^3)$ is now different being given by

$$g^2 F^{SU(2)}(\mathbf{\Gamma}) \;=\; -\frac{\mathbf{\Gamma}^2}{3} - \frac{3g^2 \ln|\mathbf{\Gamma}|}{|\mathbf{\Gamma}|} + ... \tag{10}$$

The extra factor -3 compared to (8) is the same as for β function in $4D$ non-Abelian $SU(2)$ SYM theory compared to that in 4-dimensional SQED. This is not accidental. The problem of constructing the effective lagrangian in lower dimensions and the problem of renormalization of $4D$ theories are closely related to each other. After all, renormalization is none other than contruction of effective Wilsonean lagrangian obtained after integrating out high-frequency modes. See Refs.[7, 8, 1] for more details.

Consider now an arbitrary simple, compact Lie group. The classical potential energy vanishes when $[A_j, A_k] = 0$, which implies that A_j lies in the Cartan subalgebra (and is effectively Abelian). This gives $3r$ bosonic variables in the effective lagrangian. They are supplemented by $2r$ Abelian gluino variables. These variables are organized in r superfields $\mathbf{\Gamma}^{A=1,\dots,r}$ defined as in Eqs. (4), (5). The effective lagrangian has the form $\int d^4\theta F(\mathbf{\Gamma}^A)$, where [9]

$$g^2 F(\mathbf{\Gamma}^A) \;=\; -\sum_j \left[\frac{2}{3c_V}\left(\mathbf{\Gamma}^{(j)}\right)^2 + \frac{3g^2}{|\mathbf{\Gamma}^{(j)}|}\ln|\mathbf{\Gamma}^{(j)}|\right] \tag{11}$$

with $\mathbf{\Gamma}^j = \alpha_j(\mathbf{\Gamma}^A)$ and α_j are the positive roots of the corresponding Lie algebra.

The same program can be carried out for SQM models obtained by dimensional reduction from $\mathcal{N}=2$ $4D$ theories. Consider first Abelian theory. $\mathcal{N}=2$ SQED has the same charged matter content as $\mathcal{N}=1$ theory, but involves an extra neutral chiral multiplet Φ. The lowest component of Φ gives two extra degrees of freedom in the vacuum moduli space, which thereby becomes 5–dimensional.[2] The vector superfield V and the chiral superfield Φ can be unified in a single $\mathcal{N}=4$ (in SQM sense) harmonic gauge superfield and the effective lagrangian can be formulated in the terms of the latter [10]. We use here a more conventional approach, using $\mathcal{N}=2$ superfields. The effective action depends on $\Gamma_J = (\mathbf{\Gamma}, \sqrt{2}\,\mathrm{Re}\{\Phi\}, \sqrt{2}\,\mathrm{Im}\{\Phi\})$ (interpreted as

[2] The moduli can be represented as spatial components of the gauge potential in $6D$ SQED, from which the $\mathcal{N}=2$ $4D$ theory is obtained by dimensional reduction.

superconnection in the "grandmother" $6D$ theory) and must have the form

$$S = \int dt \int d^2\theta d^2\bar{\theta}\ \mathcal{K}(\mathbf{\Gamma}, \bar{\Phi}, \Phi)\,. \tag{12}$$

Now $\mathcal{N}=2$ symmetry is manifest here. The action (12) is invariant under certain *additional* $\mathcal{N}=2$ supersymmetry transformations provided that

$$\frac{\partial^2 \mathcal{K}}{\partial \Gamma_k^2} + 2\frac{\partial^2 \mathcal{K}}{\partial \bar{\Phi}\partial \Phi} \equiv \frac{\partial^2 \mathcal{K}}{\partial \Gamma_J^2} \ = \ 0\ , \tag{13}$$

i.e. $\mathcal{K}$ is a 5–dimensional harmonic function [11]. It is convenient to unify $\mathbf{A}$ and $\phi, \bar{\phi}$ in a single 5-dimensional vector A_J and two spinors from the multiplets Γ_k and Φ in a single 4–component complex spinor η_α lying in the fundamental (spinor) representation of $SO(5) \equiv Sp(4)$. The action (12) with $\mathcal{K}$ satisfying the constraint (13) describes what can be called symplectic sigma model of the second kind. The bosonic part of the lagrangian describes motion over a 5–dimensional conformally flat manifold with the metric $h(\mathbf{A}) = -(1/2)\partial^2\mathcal{K}/\partial\mathbf{A}^2$ ($\mathbf{A}$ is 3–dimensional). It happens to be the same as for $\mathcal{N}=1$ SQED.

In non-Abelian $\mathcal{N}=2$ SYM theory with $SU(2)$ gauge group the effective action also depends on a set of $\mathcal{N}=2$ SQM superfields Γ_J, only the loop correction to the metric involves the factor -2. This is also the factor relating the one-loop charge renormalization in the $SU(2)$ $\mathcal{N}=2$ $4D$ SYM theory to that in SQED. For an arbitrary gauge group, the effective action depends on r such sets. The explicit form of $\mathcal{K}(\Gamma_J^A)$ was found in [12].

3 $D = 2$: Kähler and Twisted Models

Consider first $\mathcal{N}=1$ Abelian theory. As was noted previously, in two dimensions we have two, rather than three, moduli, representing the components of the gauge potential in the reduced dimensions. Adding superpartners, we obtain two gauge-invariant superconnections $\Gamma_{j=1,2}$. Consider the superfield $\Sigma = (\Gamma_1 + i\Gamma_2)/\sqrt{2}$. From the definition (6) and the $2D$ anticommutation relations between D_α and $\bar{D}_{\dot{\alpha}}$, we deduce that Σ satisfies the constraints

$$\bar{\mathcal{D}}_1\Sigma \ = \ \mathcal{D}_2\Sigma = 0 \tag{14}$$

and represents a so-called *twisted* chiral multiplet. It differs from the standard one by a pure convention: interchanging θ_2 and $\bar{\theta}^2$, we arrive at the standard chiral supermultiplet Φ . The effective lagrangian has the form $\mathcal{L} = \int d^4\theta \mathcal{K}$, where the Kähler potential $\mathcal{K}$ can be chosen as

$$e^2\mathcal{K}(\bar{\Phi}, \Phi) \;=\; \bar{\Phi}\Phi + \frac{e^2}{4\pi} \ln\Phi \ln\bar{\Phi} \; . \tag{15}$$

This gives a nontrivial metric on the moduli space $ds^2 = dA_j^2[1 + e^2/(2\pi A_j^2)$.

The non-Abelian generalization is straightforward. For $SU(2)$ theory, the Kähler potential has the same form as (15) with $e^2 \to g^2$ and the factor -3 multiplying the second term. For a generic gauge group, the Kähler potential represents a sum over the roots,

$$g^2\mathcal{K}(\Phi^A) \;=\; \sum_j \left[\frac{2}{c_V} \bar{\Phi}^{(j)}\Phi^{(j)} - \frac{3g^2}{4\pi} \ln\bar{\Phi}^{(j)} \ln\Phi^{(j)} \right] , \tag{16}$$

like the $1D$ prepotential in Eq. (11) ($\Phi^{(j)} = \alpha_j(\Phi^A)$).

Consider now the $\mathcal{N} = 2$ case. We start again by analyzing the Abelian theory. The effective lagrangian now involves four gauge-invarian superconnections, which can be "organized" into two complex superfields

$$\Sigma = (\Gamma_1 + i\Gamma_2)/\sqrt{2}, \qquad \Phi = (\Gamma_4 + i\Gamma_5)/\sqrt{2} \; , \tag{17}$$

where Φ is a standard chiral superfield, while Σ is a twisted one. When *both* standard and twisted multiplet are present, one cannot "untwist" the model by redefining θ and we are dealing with a genuinely *twisted* [13] σ model. A generic lagrangian can be written as

$$\mathcal{L} \;=\; \int d^2\theta d^2\bar{\theta} \; \mathcal{K}(\bar{\Phi}, \Phi; \bar{\Sigma}, \Sigma) \; . \tag{18}$$

$\mathcal{N} = 2$ supersymmetry (in $2D$ sense) is seen explicitly. One can be convinced that it is invariant under certain extra SUSY transformations, mixing Φ and Σ provided the prepotential $\mathcal{K}$ satisfies the harmonicity condition,

$$\frac{\partial^2\mathcal{K}}{\partial\bar{\Sigma}\partial\Sigma} + \frac{\partial^2\mathcal{K}}{\partial\bar{\Phi}\partial\Phi} \;=\; 0 \; . \tag{19}$$

In our case [14, 12],

$$e^2\mathcal{K} \;=\; \bar{\Sigma}\Sigma - \bar{\Phi}\Phi \; + \frac{e^2}{4\pi} \left[F\left(\frac{\bar{\Sigma}\Sigma}{\bar{\Phi}\Phi} \right) - \ln\Phi \ln\bar{\Phi} \right] , \tag{20}$$

where $F(\eta) = \int_1^\eta \ln(1+\xi)\, d\xi/\xi$ is the Spence function.

The bosonic part of the lagrangian has in this case an interesting form. Besides the standard σ model kinetic term $h(\partial_\mu\sigma\partial_\mu\bar\sigma + \partial_\mu\phi\partial_\mu\bar\phi)$, it involves also the twisted term

$$\mathcal{L}^{\rm twisted} = -\frac{e^2}{4\pi(\bar\sigma\sigma + \bar\phi\phi)}\left[\frac{\sigma}{\bar\phi}\epsilon_{\alpha\beta}(\partial_\alpha\bar\sigma)(\partial_\beta\bar\phi) + \frac{\bar\sigma}{\phi}\epsilon_{\alpha\beta}(\partial_\alpha\sigma)(\partial_\beta\phi)\right] . \tag{21}$$

This resolves the apparent conflict with the theorem [6], which normally requires the metric of a $\mathcal{N}=4$ sigma model to be hyper–Kähler. This is true only in the absence of the twisted term, and in our case the metric need not be (and is not) hyper–Kähler, and not even Kähler.

Again, the non-Abelian effective lagrangian for the $SU(2)$ theory is obtained directly from (20) by changing $e^2 \to g^2$ and introducing the factor -2 in the second term. A generic non-Abelian effective lagrangian depends on r sets $\Sigma^{(j)}, \Phi^{(j)}$ and represents a sum over the roots, like in Eqs.(11,16).

4 $D = 3$: Kähler and Hyper–Kähler Models

The effective lagrangian for $3D$ $\mathcal{N} = 2$ (in $3D$ sense) SQED depends on only one gauge-invariant superconnection in the reduced dimension Γ_3. Its component expansion (in Wess–Bagger notation) is

$$\Gamma_3 = A_3 - \frac{1}{2}\epsilon_{\mu\rho\alpha}F_{\mu\rho}\,\theta\sigma_\alpha\bar\theta \tag{22}$$

$$- D\theta\sigma_3\bar\theta + \frac{1}{4}(\partial^2 A_3)\theta^2\bar\theta^2 + \text{fermion terms} , \tag{23}$$

where $F_{\mu\rho}$ is the $3D$ electromagnetic field ($\mu,\rho = 0,1,2$). The bosonic terms in the effective lagrangian are

$$\mathcal{L} = \int \mathcal{F}(\Gamma_3)d^4\theta = h(A_3)\left[\frac{1}{2}(\partial_\mu A_3)^2 - \frac{1}{4}F_{\mu\rho}F_{\mu\rho} + \frac{D^2}{2}\right] , \tag{24}$$

where $h = -\mathcal{F}''/2$.

It is convenient at this stage to perform a *duality transformation.* To this end, write the functional integral corresponding to the lagrangian (24) in the form

$$\int \prod dF\, d\Psi\, \exp\left\{i\int d^3x\left(\mathcal{L} + \frac{1}{2}\epsilon_{\mu\rho\alpha}F_{\mu\rho}\partial_\alpha\Psi\right)\right\} . \tag{25}$$

Integrating this over $\prod d\Psi$ brings about the Bianchi constraints $\epsilon_{\mu\rho\alpha}\partial_\alpha F_{\mu\rho} = 0$, which are solved by the standard relation $F_{\mu\rho} = \partial_{[\mu}A_{\rho]}$. But let us instead do the integral in Eq. (25) over $\prod dF$ first. We are left with

$$\prod d\Psi \exp\left\{ i \int d^3x \left[\frac{h}{2}(\partial_\mu A_3)^2 + \frac{1}{2h}(\partial_\mu \Psi)^2 \right] \right\} . \tag{26}$$

The integrand in the exponent is the dual lagrangian (the bosonic part thereof). The scalar field Ψ is the dual photon.

Let us now introduce the field

$$B \;=\; -\frac{\mathcal{F}'(A_3)}{2}, \tag{27}$$

so that $\partial_\mu B = h\,\partial_\mu A_3$. Introducing a complex variable $\phi = (B + i\Psi)/\sqrt{2}$, we can write $\mathcal{L}_{\rm dual}$ in the Kähler form $\int d^4\theta\, \mathcal{K}(\bar\Phi, \Phi)$. The relation between the Kähler potential $\mathcal{K}$ and the function $\mathcal{F}$ can be inferred from Eq. (27).

For the effective lagrangian of $3D$ SQED, the particular form of the Kähler potential is

$$\mathcal{K} \;=\; e^2 \left[\Delta^2 + \frac{1}{2\pi}\Delta \ln \Delta \right] , \tag{28}$$

where $\Delta = (\bar\Phi + \Phi)/\sqrt{2}$. It gives the metric

$$e^2 h_{3D} \;=\; 1 + \frac{e^2}{4\pi |A_3|} + \ldots \tag{29}$$

With the experience acquired from the analysis of the $D = 1$ and $D = 2$ theories, these expressions can be easily generalized to the non–Abelian case.

For $D = 3$, this is not yet the end of the story. Considerations of supersymmetry alone do not exclude the presence of a *superpotential* $\sim \mathrm{Re} \int d^2\theta\, F(\Delta^A)$ on top of the Kähler potential in the effective lagrangian. Indeed, such a superpotential is generated in non-Abelian $3D$ theories by a *non-perturbative* mechanism [15]. The mechanism is roughly the same as the known instanton mechanism for generating a superpotential in $4D$ $\mathcal{N} = 1$ SYM theory with matter [16]. In three dimensions, instantons are t' Hooft–Polyakov monopoles. They have two fermion "legs" (zero modes) which lead to generation of gluino condensate. The superpotential can be recovered from the condensate. In the simplest $SU(2)$ case, it has the form

$$F(\Delta) \;\sim\; g^4 \exp\left\{ -2\sqrt{2}\pi\Phi \right\} . \tag{30}$$

The superpotential (30) lifts the degeneracy of the valley. Actually, the scalar potential $U \sim \exp\left\{-2\sqrt{2}\pi(\phi+\bar{\phi})\right\}$ corresponding to the superpotential (30) (the exponent $2\sqrt{2}\pi(\phi+\bar{\phi}) = 4\pi A_3/g^2$ is nothing but the $3D$ instanton action) does *not* have a minimum at a finite value of ϕ (as it does not in the massless $\mathcal{N}=1$ supersymmetric QCD — this is a typical "runaway vacuum" phenomenon). In 4-dimensional SQCD, this can be cured by giving a mass to the matter fields: the supersymmetric vacuum would then occur at a finite value of ϕ. But in the framework given here, the form of the lagrangian of the $3D$ is fixed by the original $4D$ theory and we have to conclude that the $3D$ $\mathcal{N}=1$ non-Abelian sister simply does not exist as a consistent theory.

The effective lagrangian for $\mathcal{N}=4$ $3D$ SYM theory, involves $4r$ moduli: three components of the vector-potential in reduced dimensions $\mathbf{A}$ and a dual photon for each unit of the rank. In this case, the effective lagrangian has a form of conventional hyper-Kähler σ model [17]. In the $SU(2)$ case, the metric describes the Atiyah–Hitchin manifold [18]. An analytic expression for the metric exists, but it is not so simple, involving elliptic functions, etc. But for large values of $|\mathbf{A}|$, the expression is greatly simplified and acquires the form

$$ds^2 \;=\; \left(1-\frac{g^2}{2\pi|\mathbf{A}|}\right)d\mathbf{A}^2+\frac{\left(d\Psi+\frac{g^2}{2\pi}\omega\right)^2}{\left(1-\frac{g^2}{2\pi|\mathbf{A}|}\right)}\;. \tag{31}$$

The dual variable Ψ describes the dual photon. $\omega(\mathbf{A})$ is the 1-form describing the Abelian connection of a Dirac monopole and can be chosen as $\omega(\mathbf{A}) = \cos\theta d\phi$. Mathematically, this is the so called Taub-NUT metric with negative mass term. Physically, the approximation (31) corresponds to neglecting all the non-perturbative instanton-driven terms and retaining only the perturbative one-loop contribution. This expression can be obtained from $2D$ effective lagrangian (and the latter from the $1D$ one) by "unfolding the ring" procedure. We refer the reader to [19, 1] for detailed explanations.

A remarkable fact is that the Atiyah-Hitchin metric arises also in a completely different problem, describing the low–energy dynamics of two BPS monopoles [20]. In this case, the vector $\mathbf{A}$ acquires the meaning of the monopole separation $\mathbf{r}$ and Ψ of their relative phase. The classical trajectories of the monopoles represent geodesics on the AH manifold. The approximation (31) corresponds to the region when the distance between monopoles is much larger than the size of their cores.

An analog of (31), taking into account only perturbative contributions, can be written for an arbitrary gauge group. The effective lagrangian is

$$g^2\mathcal{L} \ = \ \frac{1}{2}(\partial_\mu \mathbf{A}^A)(\partial_\mu \mathbf{A}^B)Q_{AB} + \frac{1}{2}J_\mu^A Q^{-1}_{AB} J_\mu^B \ , \tag{32}$$

where

$$\begin{aligned} Q_{AB} &= \delta_{AB} - \frac{g^2}{2\pi}\sum_j \frac{\alpha_j^A \alpha_j^B}{|\mathbf{A}^{(j)}|} \ , \\ J_\mu^A &= \partial_\mu \Psi^A + \frac{g^2}{2\pi}\sum_j \boldsymbol{\omega}(\mathbf{A}^{(j)})\partial_\mu \mathbf{A}^{(j)}\alpha_j^A \ , \end{aligned} \tag{33}$$

$\mathbf{A}^{(j)} = \alpha_j(\mathbf{A}^A) \equiv \alpha_j^A \mathbf{A}^A$.

These are asymptotic expressions for the metric. They involve singularities and their structure is complicated. However, re-summation of instanton corrections should patch up these singularities. The result of such a re-summation gives a smooth hyper–Kähler manifold.

The simplest case are the unitary groups [21]. The Cartan subalgebra of $SU(N)$ consists of traceless $N \times N$ diagonal matrices. As far as the effective lagrangian (32) is concerned, we have four such matrices: diag $(\mathbf{A}_1, \ldots, \mathbf{A}_N)$ and diag $(\Psi_1, \ldots, \Psi_N)$, $\sum_m \mathbf{A}_m = \sum_m \Psi_m = 0$. There are $N(N-1)/2$ positive roots, $\alpha_{ml}(\mathbf{A}) = \mathbf{A}_m - \mathbf{A}_l, m < l = 1, \ldots, N$. Substituting it in Eq. (32), we obtain the generalized Taub-NUT metric describing the dynamics of N well–separated BPS monopoles [20]. $\mathbf{A}_m \equiv \mathbf{r}_m$ and Ψ_m are interpreted as the positions and phases of individual monopoles. The condition $\sum_m \mathbf{r}_m = 0$ (and similarly for phases) means that the trivial center of mass motion is separated out.

The metric thus obtained is singular for certain small values of the distances between the monopoles $|\mathbf{r}_m - \mathbf{r}_l|$. These singularities can be patched, however, and with all probability *are* patched by the instanton corrections. A conjecture of existence and uniqueness can now be formulated: there is only one smooth hyper-Kähler manifold of dimension $4(N-1)$ (a *generalized* Atiyah–Hitchin manifold) with the given generalized Taub-NUT asymptotics. (I bet there is, though, as far as I know, this has not yet been proven mathematically in an absolutely rigorous way.) An explicit expression for the generalized AH metric is not known.

Also for the other classical groups, the metric (32) can be interpreted in the monopole terms, but generically, certain constraints should be imposed.

For example, for the symplectic groups $Sp(2r)$, the effective Lagrangian is associated with the dynamics of $2r + 1$ BPS monopoles numbered by the integers $j = -r, \ldots, r$, with imposing the constraints

$$\begin{aligned} \mathbf{r}_{-r} + \mathbf{r}_r \ &= \ldots = \mathbf{r}_{-1} + \mathbf{r}_1 \ = \ 2\mathbf{r}_0 \ = \ 0 \ , \\ \Psi_{-r} + \Psi_r \ &= \ldots = \Psi_{-1} + \Psi_1 \ = \ 2\Psi_0 \ = \ 0 \ . \end{aligned} \tag{34}$$

We *are* allowed to impose these constraints because they are compatible with the classical equations of motion (this nontrivial fact can be checked explicitly). A similar contruction can be made for other classical groups. We arrive at hyper–Kähler manifolds of a new kind, not considered by mathematicians before. They are obtained from the generalized Atiyah-Hitchin manifolds by the procedure of *hyper-Kähler reduction* [22] involving (not for $Sp(2r)$, but for other groups) also certain deformations. The number of monopoles to start with concides with the dimension of the minimal representation in the corresponding group. For example, for E_8, we have to take 248 monopoles.

To conclude with, we present a table where all the members of the family of effective theories considered above are shown.

	$\mathcal{N} = 1$	$\mathcal{N} = 2$
$D = 1$	Symplectic σ model of the first kind	Symplectic σ model of the second kind
$D = 2$	Kähler σ model	Twisted σ model (GHR)
$D = 3$	Kähler σ model with superpotential. Runaway vacuum	Hyper–Kähler σ model
$D = 4$	No moduli space. Discrete vacua	SW effective theory

Table 1: Pure SYM: the family of effective theories.

References

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