Noncommutative U(1) gauge model in axial gauge.

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Abstract

The problem of UV-IR mixing in noncommutative quantum gauge invariant models is discussed. A gauge invariant $U(1)$ model on the noncommutative spacelike plane is presented and analyzed in the axial gauge. It is shown to be free of nonintegrable infrared singularities.

1 Introduction

In this talk I discuss some progress in understanding of quantum noncommutative gauge invariant theories, based on the results, presented in my papers ([1], [2]).

The most striking feathure of quantum noncommutative models is appearance of nonintegrable infrared singularities in radiative corrections to scattering amplitudes. Some UV-finite one loop diagrams have singularities at zero external momenta. That leads to a drastic modification of the dispersion law which makes a physical interpretation of the corresponding states rather cumbersome. Moreover, when diagrams having infrared poles are inserted as subdiagrams to multiloop graphs the corresponding integrals acquire nonintegrable infrared singularities. As a result quantum noncommutative models have not only UV-singularities which may be removed with the help of a proper renormalization, but also infrared singularities which make these models inconsistent ([3]- [12]). Infrared poles arise in conditionally convergent diagrams, which by power counting have the divergency index equal to 1 or 2. It makes plausible that such poles are absent in renormalizable supersymmetric models, where in the commutative case all UV-divergencies are logarithmic by power counting. Indeed explicit calculations of one-loop diagrams in $N = 1$ supersymmetric gauge models showed that in this case infrared pole singularities are absent ([13]-[15]). If this property holds in higher loop diagrams as well, supersymmetric noncom- mutative gauge models would allow a consistent treatment. At present the consistency was proven to all orders for supersymmetric Wess-Zumino model ([16]) and for the three dimensional supersymmetric $U(1)$ model ([17]). Nevertheless such models are hardly satisfactory as a smooth commutative limit does not exist and radiative corrections in these models produce large Lorentz symmetry breaking effects.

Recently we proposed a modification of the nonsupersymmetric $U(1)$ model on the spacelike noncommutative plane, which seems to be free of nonintegrable infrared singularities ([1]). Contrary to the straight forward generalization of QED, this model describes a spin zero particle. It has a smooth commutative limit, so Lorentz symmetry breaking effecta are under control and may be done small. In the following sections we shall describe in more details the problems related to UV-IR mixing in noncommuta- tive gauge theories and possible ways to resolve these problems.

2 Noncommutativity and UV-IR mixing

Last years noncommutative models attracted a new interest, mainly in connection with development of string and matrix models. For example, it was shown ([18]) that the supersymmetric gauge theory on a noncommutative torus is related to a compactification of the matrix model with the action

$$S = \int dttr(\sum_{i,j}(D_i X_j)^2 - \sum_{i<j}[X_i, X_j]^2) \tag{1}$$

where X_j are t-dependent matrices and D is a covariant derivative

$$D = \partial_t + iA_0(t) \tag{2}$$

Another important application of noncommutative theories was found by Seiberg and Witten ([19]), who showed that in a certain limit the dynamics of the string model with a nonzero magnetic field may be described by a noncommutative super- symmetric gauge theory.

In these and other examples noncommutative models appear as some effective theories describing some more fundamental models for certain values of parameters.

There are also attempts (both theoretical and experimental) to consider noncom- mutative models as fundamental ones, leading to small violations of Lorentz invari- ance at very small distances. In the classical theory choosing the parameter ξ in the eq.(3) small enough, one can supress corrections to the usual commutative models avoiding a contradiction with experiment. However such approach make sense only if the quantum corrections are also negligible for small values of the noncommutativity parameter ξ. Similar problems arise when noncommutative models are considered as effective theories: in all these cases the quantum corrections must be under control and should not change drastically classical results.

It appears that in noncommutative models new divergencies, assosiated with the infrared behaviour of matrix elements arise, which make questionable their selfconsistency.

Let us consider this phenomenon in more details.

The noncommutative $U(1)$ model is described by the action

$$S = \int d^4x\{-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}\} \tag{3}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]* \tag{4}$$

The symbol $[]*$ means the commutator in which the star product is used. The star product is defined as follows

$$f(x) * g(x) = \exp\{i\xi\theta_{\mu\nu}\partial^x_\mu\partial^y_\nu\}f(x)g(y)_{y=x} \tag{5}$$

where $\theta_{\mu\nu}$ is a real antisymmetric matrix and ξ is a noncommutativity parameter. In the limit $\xi \to 0$ the action (3) obviously reduces to the free electromagnetic action.

The gauge transformations look similar to nonabelian gauge transformations

$$\delta A_\mu = \partial_\mu\epsilon - ig(A_\mu * \epsilon - \epsilon * A_\mu) \tag{6}$$

Note that although for a general skewsymmetric matrix $\theta_{\mu\nu}$ the interaction (3) is nonlocal, models with $\theta_{i0} = 0$ introduce only spatial nonlocality and the standard Hamiltonian formalism may be applied. In what folows we assume that $\theta_{i0} = 0$ and Hamiltonian formalism may be used. We take $\theta_{12} = -\theta_{21} = 1; \theta_{13} = \theta_{23} = 0$.

In this case the quantization may be done in a standard way, and the Feynman rules look similar to the usual Yang-Mills theory. In the covariant α-gauges the Faddeev-Popov ghosts appear with the Lagrangian

$$L = \partial_\mu \bar{c}(\partial_\mu c - ig[A_\mu, c]*) \tag{7}$$

The quadratic part of action in the noncommutative case coincides with the correspon- ding part of the commutative model, hence the propagators have the same form. However the interaction looks differently. The main difference is the appearance of oscillating factors in the interaction vertices. The three point gauge vertex with the momenta k_1, k_2, k_3 and indices μ, ν, ρ has a form

$$2ig\sin(\xi k_1 \tilde{k_2})[(k_1 - k_2)_\rho \delta_{\mu\nu} + (k_2 - k_3)_\mu \delta_{\nu\rho} + (k_3 - k_1)_\nu \delta_{\mu\rho}] \tag{8}$$

and the four point vertex is

$$-4g^2[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})\sin(k_1\tilde{k_3}\xi)\sin(k_2\tilde{k_4}\xi) + sym] \tag{9}$$

where sym means symmetrization with respect to interchange of momenta k_i and indices μ, ν, ρ. Here we used the notation $\tilde{k_\mu} = \theta_{\mu\nu}k_\nu$.

Using these Feynman rules one can write the analytic expression for the one loop gauge boson polarization operator. In dimensional regularization it looks as follows (up to some unimportant finite terms):

$$\Pi_{\mu\nu}(p) = \frac{4g^2\mu^{2\epsilon}(2\pi)^d \int d^dk \sin^2(k\tilde{p})(-\frac{3}{k^2}delta_{\mu\nu} - \frac{k_\mu k_\nu + k_\mu}{p}{}_\nu}{k^2(p+k)^2} +$$
$$\frac{k^2 + (k+p)^2 + 4p^2)\delta_{mu\nu} + 10k_\mu k_\nu - 2p_\mu p_\nu + 5k_\mu p_\nu + 5p_\mu k_\nu}{2k^2(k+p)^2} \tag{10}$$

To get an idea what kind of singularities one may expect, let us consider the first term in the integral (10) in more details. We rewrite it in the form

$$I = \int d^dk \frac{1}{2}(1 - \cos(2k\tilde{p})\frac{3}{k^2} \tag{11}$$

First of all we see, that the presence of trigonometric factors does not remove ultraviolet infinities completely. The integral I consists of two terms. The first one is quadratically divergent and the divergency may be removed by introduction of the usual counterterms. (In the case under consideration this counterterm would correspond to the photon mass renormalization. Due to the gauge invariance of the model in dimensional regularization this term is absent). The other terms in the integral (10) include logarithmically divergent parts, which require the wave function renormalization counterterm. The most interesting is the second term in the integral (11). This term is convergent and can be calculated explicitely:

$$I = \frac{\pi^2}{\Gamma(1)}(\tilde{p})^{-2} \tag{12}$$

One sees that the integral (12) is singular at $\tilde{p}^2 = 0$. The origin of this singulariry is easy to understand. The second term in the integral (11) is convergent for any $\tilde{p} \neq 0$ due to the presence of the oscillating factor. However for $\tilde{p} = 0$ oscillations are absent and the integral diverges quadratically. This is the phenomenon of UV-IP mixing, which is characteristic for noncommutative models.The oscillating factors in the interaction vertices supress some of the UV-divergencies, but at the same time result in the appearance of infrared singularities. In particular the gauge field polarization operator has ultraviolet divergent part corresponding to so called planar diagrams, and the convergent nonplanar part, which contains the term singular at $p = 0$. Explicit calculation gives

$$\Pi_{\mu\nu}(p) = \frac{g^2}{2\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{\xi^2(\tilde{p}^2)^2} + \dots \tag{13}$$

where ... denotes the terms proportional to $\ln(\xi p^2)$ and terms, regular at $p = 0$.

Similar singularities appear in the three point function, which looks as follows

$$\Gamma_{\mu\nu\rho}(p,q) \sim \cos(\xi p\tilde{q})\{\frac{\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho}{\xi(\tilde{p}^2)^2} + sym\} + \dots \tag{14}$$

where ... again stands for less singular terms and sym means symmetrization

$$p \to q, \mu \to \nu; \quad p \to -(p+q), \mu \to \rho; \tag{15}$$

It is worth to notice that the singular terms (14, 15) are compatible with the gauge invariance. Ward identity for the polarization operator is equivalent to transversality of $\Pi_{\mu\nu}(p)$. As $p_\mu\tilde{p}_\mu = 0$, this condition is obviously satisfied.

The singularity of $\Pi_{\mu\nu}$ at $p = 0$ changes drastically the dispersion relation. A new singularity at $p = 0$ appear. Moreover, when the diagrams having pole singulaities at $p = 0$ are inserted as subgraphs ro more complicated diagrams, they may generate nonintegrable infrared singularities, making the theory inconsistent.

All other diagrams are either regular at zero momenta or have logarithmic infrared singularities. In general infrared pole singularities arise in the diagrams which in the absence of phase factors would be quadratically or linearly ultraviolet divergent. Logarihmically divergent diagrams produce only logarithmic infrared singularities which do not spoil integrability. Although in the classical theory noncommutativity may lead to tiny deviations from the commutative case, provided the parameter ξ is small, in the quantum theory in no way noncommutativity may be considered as a small effect.

In the paper ([1]) we suggested that the presence of nonrenormalizable divergen- cies in the noncommutative $U(1)$ model indicates that this theory is not complete and the classical action should be modified by adding new gauge invariant terms. As we have seen, the new singular structures appearing in radiation corrections are proportional to $\tilde{p}$. So it is natural to look for a new action which contains such terms from the very beginning. The other natural requirements which we impose are the following: the modified action must be gauge invariant, introduce nonlocality only via Moyal products and in the limit $\xi \rightarrow 0$ describe a Lorentz invariant theory.

The classical action proposed in ([1]) looks as follows

$$A = \int d^4x \{ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \lambda(x) \theta_{\mu\nu} F_{\mu\nu}(x) \} \tag{16}$$

Here $\lambda(x)$ is the Lagrange multiplier transforming according to adjoint represen- tation of the gauge group.

At first site the action (16) breaks Lorentz invariance even in the commutative limit. However a proper hamiltonian analysis shows that in the limit $\xi \rightarrow 0$ Lorentz invariance is restored.

Variation of the action (16) over $\lambda(x)$ produces the new constraint:

$$\theta_{\mu\nu} F_{\mu\nu}(x) = 0 \tag{17}$$

leading to the reduction of the number of physical degrees of freedom. Although the action (16) is written in terms of the vector field A_μ, it describes a spin zero particle. In the next section we shall show that fixing a gauge in a proper way and rescaling variables, one can rewrite the theory in the form when Lorentz invariance in the commutative limit is manifest. Moreover, we shall prove that radiative corrections for the modified models do not generate infrared pole singulariries and the limit $\xi \to 0$ exists at any order of perturbation theory.

3 The modified U(1) model in axial gauge.

The easiest way to demonstrate the absence of infrared poles and existence of a Lorentz invariant commutative limit is to consider the model in the axial gauge $A_1 = 0$ ([2]). In this gauge the constraint (17) reduces to

$$\partial_1 A_2 = 0 \tag{18}$$

where we took into account that the only nonzero elements of the matrix $\theta_{\mu\nu}$ are θ_{12} and θ_{21}. Assuming the usual asymptotic condition $A_\mu(x) \to 0$ when $\vec{x} \to \infty$ we see, that the eq.(18) implies $A_2 = 0$.

The classical action (16) in the chosen gauge has a form

$$A = \frac{1}{2}(\partial_0 A_3 - \partial_3 A_0 + ig[A_0, *A_3])^2 + \\ + \frac{1}{2}\sum_{i=1,2}(\partial_i A_0)^2 - \frac{1}{2}\sum_{i=1,2}(\partial_i A_3)^2 \tag{19}$$

Introducing the canonical momentum

$$p_3 = \frac{\partial L}{\partial \dot{A}_3} = F_{03} \tag{20}$$

we may write the action in the form

$$A = \int \{p_3 \dot{A}_3 - \frac{1}{2}p_3^2 + A_0 D_3 p_3 + \frac{1}{2}\sum_{i=1,2}(\partial_i A_0)^2 - \frac{1}{2}\sum_{i=1,2}(\partial_i A_3)^2\} dx \tag{21}$$

Here D_3 is the covariant derivative. The variable A_0 can be excluded by solving the corresponding time independent equation

$$A_0 = (\nabla)^{-2} D_3 p_3, \quad (\nabla)^2 = \partial_1^2 + \partial_2^2 \tag{22}$$

Substituting this solution to the action (21) we get

$$A = \int \{p_3 \dot{A}_3 - \frac{1}{2}p_3^2 - \frac{1}{2}D_3 p_3 \nabla^{-2} D_3 p_3 - \frac{1}{2}\sum_{i=1,2}(\partial_i A_3)^2\} dx \tag{23}$$

This equation describes a scalar particle with the Hamiltonian

$$H = \frac{p_3^2}{2} + \frac{1}{2} D_3 p_3 \nabla^{-2} D_3 p_3 - \frac{1}{2} \sum_{i=1,2} (\partial_i A_3)^2 \tag{24}$$

and the canonical quantization is applied directly.

The action (16) is not invariant with respect to the Lorentz transformation, however in the classical commutative limit $\xi \to \infty$ the invariance of the theory is restored.

Indeed in the commutative limit the action acquires a form

$$A = \int \{p_3 \dot{A}_3 - \frac{1}{2} p_3^2 - \frac{1}{2} \partial_3 p_3 \nabla^{-2} \partial_3 p_3 - \frac{1}{2} \sum_{i=1,2} (\partial_i A_3)^2\} dx \tag{25}$$

In terms of the rescaled variables

$$\pi(k) = [1 + k_3^2 (k_1^2 + k_2^2)^{-1}]^{\frac{1}{2}} \tilde{p}_3(k) \tag{26}$$

$$\phi(k) = [1 + k_3^2 (k_1^2 + k_2^2)^{-1}]^{-\frac{1}{2}} \tilde{A}_3(k) \tag{27}$$

it is the standard Lorentz invariant action describing the free scalar particle:

$$A = \int \{\pi \dot{\phi} - \frac{1}{2} \sum_{i=1}^{3} (\partial_i \phi)^2\} dx \tag{28}$$

However the quantum corrections may break this smooth transition as it happens in the standard noncommutative $U(1)$ model. Below we shall show that in our model the smooh transition to the free scalar particle exists in the quantum case as well. All quantum corrections vanish in the limit $\xi \to 0$.

The scattering matrix may be presented as usual in the form of the path integral of the $\exp\{iA\}$, where A is given by the eq.(23). It is more convenient to write it introducing the integration over A_0:

$$S = \int \exp\{i \int [p_3 \dot{A}_3 - \frac{1}{2} p_3^2 + A_0 D_3 P_3 + \\ + \frac{1}{2} \sum_{i=1,2} (\partial_i A_0)^2 - \frac{1}{2} \sum_{i=1,2} (\partial_i A_3)^2] dx\} dA_3 dA_0 \tag{29}$$

where the usual asymptotic conditions for A_3 are assumed.

Performing the Gaussian integration over P_3 we get

$$S = \int \exp\{i \int [\frac{1}{2}F_{03}^2 + \frac{1}{2}\sum_{i=1,2}(\partial_i A_0)^2 - \frac{1}{2}\sum_{i=1,2}(\partial_i A_3)^2]dx\}dA_3 dA_0 \qquad (30)$$

The propagator is determined by the inverse of the quadratic form of the free action (30) and looks as follows

$$D_{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon}\{g^{\mu\nu} - \frac{k_\mu k_\nu}{k_1^2 + k_2^2}\}; \quad \mu, \nu = 0, 3 \qquad (31)$$

As all other elements of $D_{\mu\nu}$ are zero, it obviously satisfies the constraint equation

$$\tilde{k}_\mu D_{\mu\nu}(k) = 0 \qquad (32)$$

Due to gauge invariance of the action (16) the Green functions satisfy Generalized Ward Identities:

$$\int \exp\{iA + \int [\frac{1}{2\alpha}(A_1)^2 + J_\mu A_\mu]dx\}$$
$$\{\frac{1}{\alpha}\partial_1 A_1(x) - \partial_\mu J_\mu(x) + ig[A_\mu(x), *J_\mu(x)]\}dA_\mu = 0 \qquad (33)$$

For the two-point function one has

$$-\frac{i}{\alpha} < \partial_1 A_1(x) A_\nu(y) >= \partial_\nu \delta(x - y) \qquad (34)$$

Fourier transform of the polarization operator satisfies the transversality condition

$$p_\mu \Pi_{\mu\nu} = 0 \qquad (35)$$

Let us consider the polarization operator in more details. It is easy to see that the pole singular terms in this function depend only on $\tilde{p}$. Indeed, the polarization operator has a form

$$\Pi_{\mu\nu}(p) = \int \prod_i dk_i \sin(k_1 \tilde{p}\xi) \sin(k_n \tilde{p}\xi)[f_{\mu\nu}(k_1 \ldots k_n, p_0, p_3, \tilde{p})] \qquad (36)$$

Here the integrand is a rational function with the dimension p^{-4n+2} , multiplied by the product of trigonometric functions. Let us present $\Pi_{\mu\nu}(p)$ as follows

$$\Pi_{\mu\nu}(p) = \int [f_{\mu\nu}(k_1 \dots k_n, p_0, p_3, \tilde{p}) - f_{\mu\nu}(k_1 \dots k_n, 0, 0, \tilde{p})] \times \\ \times \prod_i \sin(k_1 \tilde{p}\xi)\sin(k_n \tilde{p}\xi) dk_i \\ + \int f_{\mu\nu}(k_1 \dots k_n, 0, 0, \tilde{p}) \prod_i \sin(k_1 \tilde{p}\xi)\sin(k_n \tilde{p}\xi) dk_i \quad (37)$$

The first term cannot produce the second order infrared pole, as the integral of the rational part diverges at most linearly. In fact for symmetry reasons the divergency is logarithmic. Hence the pole infrared singularities may be present only in the second term, which depends only on $\tilde{p}$.

The only possible structure compatible with the Ward identities (33) is

$$\Pi^{pole}_{\mu\nu}(p) = \tilde{p}_\mu \tilde{p}_\nu \Pi(\tilde{p}) \quad (38)$$

Now we recall that the free propagator of the field A_μ, given by the eq.(31), satisfies the transversality condition (32). Therefore a possible infrared pole singula- rity of $\Pi_{\mu\nu}(p)$ does not contribute and is irrelevant for calculation of any diagram. The only allowed infrared singularity is logarithmic $\sim \ln(\tilde{p}^2\xi)$. This singularity is integrable and does not make the theory inconsistent.

In fact in the gauge under consideration the absence of infrared pole singularities is quite obvious. The only nonzero elements of of the free propagator $D_{\mu\nu}$ correspond to $\mu, \nu = 0, 3$. At the same time the pole singular part of the polarization operator $\Pi_{\mu\nu}$ depends only on $\tilde{p}_\mu$, and hence is proportional to p_1 or p_2. Obviously

$$D_{\mu\nu}\Pi^{pole}_{\nu\alpha} = 0 \quad (39)$$

Similar arguments are applied to the three point function. By the same reason the pole singular part depends only on $\tilde{p}, \tilde{q}$ and hence

$$D_{\mu\alpha}(p)\Gamma^{pole}_{\alpha\nu\rho}(p,q) = D_{\nu\alpha}(q)\Gamma^{pole}_{\mu\alpha\rho}(p,q) = D_{\rho\alpha}(p+q)\Gamma^{pole}_{\mu\nu\alpha}(p,q) = 0 \quad (40)$$

As in the case of polarization operator the only possible infrared singularities of $\Gamma_{\mu\nu\rho}(p,q)$ are logarithmic.

All other diagrams are at most logarithmically UV divergent by power counting and cannot produce infrared pole singularities.

So we proved that the the $U(1)$ noncommutative model described by the action (16) is free from nonintegrable infrared singularities and allows calculation of radiation corrections to arbitrary order. Note that all interaction vertices in the commutative limit $\xi \to 0$ vanish as ξ. Taking into account that possible singularities in ξ are $\sim \ln(\xi)$, we conclude that the limit $\xi \to 0$ exists in the quantum version of our model and describes the free scalar particle.

Our conclusion does not change if the interaction with a spinor field in adjoint representation is included. The model is free from infrared pole singularities and in the limit $\xi \to 0$ is Lorentz invariant. If the interaction includes also spinor field in the fundamental representation infrared singularities are still absent, but in the limit $\xi \to 0$ Lorentz invariance may be broken.

4 Discussion

In this paper we considered the problem of ultraviolet-infrared mixing in quantum noncommutative gauge theories. We presented a formulation of the consistent $U(1)$ model on the noncommutative plane (x_1, x_2). Contrary to usual gauge invariant theories the gauge fields in our model correspond to spin zero particles. It is an open question if one can include in models of these kind also vector particles. At present we can say nothing about possible physical significance of the model described above. Our main goal was to demonstrate that consistent noncommutative gauge invariant models, posessing a smooth commutative limit do exist. **Acknowledgements.**
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