Nonsingular Increasing Gravitational Potential for the Brane in 6D

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Abstract

We present a new (1+3)-brane solution to Einstein equations in (1+5)-space. As distinct from previous models this solution is free of singularities. The gravitational potential transverse to the brane is an increasing (but not exponentially) function and asymptotically approaches a finite value. The solution localizes the zero modes of all kinds of matter fields and Newtonian gravity on the brane. An essential feature of the model is that different kind of matter fields have different localization distances.

The scenario where our world is associated with a brane embedded in a higher dimensional space-time [1, 2, 3] has attracted a lot of interest. In the models of [2, 3] gravitons, which are allowed to propagate in the bulk, are confined on the brane because of a warped geometry. However, there are difficulties with the choice of a natural trapping mechanism for some matter fields. For example, in the existing (1+4)-dimensional models spin 0 and spin 2 fields are localized on the brane with an exponentially decreasing gravitational warp factor, spin 1/2 field are localized with an increasing factor

[4], and spin 1 fields are not localized at all [5]. For (1+5)-dimensions spin 0, 1 and 2 fields are localized on the brane with a decreasing warp factor and spin 1/2 fields again are localized with an increasing factor [6].

The reason why there are problems with localization of fermions in warped geometries is that the fermionic Lagrangians have an increasing exponential, coming from the metric tensors with upper indices, g^{AB}, and from the tetrads, $h^{\bar{A}}_{B}$. Thus the action integral over the extra coordinates diverges, which signals the non-localization of the fermionic fields. In both (1+4)- and (1+5)-space models with warped geometry one needs to introduce some non-gravitational interaction to localize all the Standard Model particles.

For reasons of economy and to avoid charge universality obstruction [7] one would like to have a universal gravitational trapping mechanism for all fields. In [8, 9, 10] we found such a solution of the 6-dimensional Einstein equations in (2+4)- and (1+5)-spaces, which localized all kind of bulk fields on the brane. These solutions contain non-exponential scale factors, which increase from the brane, and asymptotically approach a finite value at infinity. In [11] the solution of [9] was generalized to the case of n dimensions. In this paper we present a new solution to the Einstein equations in (1+5)-space, which, similar to the models [8, 9], localizes all kind of fields, but is free of singularities. Because of this feature of the solution we are able to fix the free parameters of the model by setting realistic boundary conditions .

The general form of action of the gravitating system in six dimensions is

$$S = \int d^6x\sqrt{-^6g}\left[\frac{M^4}{2}(^6R + 2\Lambda) +^6 L\right] , \tag{1}$$

where $\sqrt{-^6g}$ is the determinant, M is the fundamental scale, ^6R is the scalar curvature, Λ is the cosmological constant and ^6L is the Lagrangian of matter fields. All of these quantities are six dimensional.

The 6-dimensional Einstein equations with stress-energy tensor T_{AB} are

$$^6R_{AB} - \frac{1}{2}g_{AB}\ ^6R = \frac{1}{M^4}\left(\Lambda g_{AB} + T_{AB}\right) . \tag{2}$$

Capital Latin indices run over $A, B, ... = 0, 1, 2, 3, 5, 6$.

As in [8, 9] we choose the ansatz for the 6-dimensional metric as

$$ds^2 = \phi^2(r)\eta_{\alpha\beta}(x^\nu)dx^\alpha dx^\beta - \lambda(r)(dr^2 + r^2 d\theta^2) , \tag{3}$$

where the Greek indices $\alpha, \beta, ... = 0, 1, 2, 3$ refer to 4-dimensional coordinates. The metric of ordinary 4-space, $\eta_{\alpha\beta}(x^\nu)$, has the signature $(+, -, -, -)$. The functions $\phi(r)$ and $\lambda(r)$ depend only on r, and thus are cylindrically symmetric in the transverse polar coordinates ($0 \leq r < \infty$, $0 \leq \theta < 2\pi$).

The stress-energy tensor T_{AB} is assumed to have the form

$$T_{\mu\nu} = -g_{\mu\nu}F(r), \qquad T_{ij} = -g_{ij}K(r), \qquad T_{i\mu} = 0 \ . \tag{4}$$

Using the ansatz (3), the energy-momentum conservation equation gives the following relationship between the source functions $F(r)$ and $K(r)$ from (4)

$$K' + 4\frac{\phi'}{\phi}(K - F) = 0 \ . \tag{5}$$

We want to point out a problem associated with the source functions (4). In general the Einstein equations have an infinite number of solutions generated by different matter energy-momentum tensors, most of which have no clear physical meaning. There is a great freedom in the choice of $F(r)$ and $K(r)$; the only restriction on their form is (5). It is not easy to construct realistic source functions from fundamental matter fields so that the brane is a stable, localized object. We shall determine $F(r)$ and $K(r)$ from the general physical assumptions that they are smooth functions of the radial coordinate r, describe a continuous matter distribution for all r, and that they decrease outside the brane $r > \epsilon$, where ϵ is the brane width.

To solve equations (2) we take the 4-dimensional Einstein equations have the ordinary form without a cosmological term *i.e.* $R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 0$. The Ricci tensor in four dimensions $R_{\alpha\beta}$ is constructed from the 4-dimensional metric tensor $\eta_{\alpha\beta}(x^\nu)$ in the standard way. With ansätze (3) and (4) the $\alpha\alpha$, rr, and $\theta\theta$ components Einstein field equations (2) respectively become

$$\begin{aligned} 3\frac{\phi''}{\phi} + 3\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} + \frac{1}{2}\frac{\lambda''}{\lambda} - \frac{1}{2}\frac{(\lambda')^2}{\lambda^2} + \frac{1}{2}\frac{\lambda'}{r\lambda} &= \frac{\lambda}{M^4}[F(r) - \Lambda] \ , \\ \frac{\phi'\lambda'}{\phi\lambda} + 2\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} &= \frac{\lambda}{2M^4}[K(r) - \Lambda] \ , \\ 2\frac{\phi''}{\phi} - \frac{\phi'\lambda'}{\phi\lambda} + 3\frac{(\phi')^2}{\phi^2} &= \frac{\lambda}{2M^4}[K(r) - \Lambda] \ , \end{aligned} \tag{6}$$

where the prime $= \partial/\partial r$.

Subtracting the rr from the $\theta\theta$ equation and multiplying by ϕ/ϕ' gives

$$\frac{\phi''}{\phi'} - \frac{\lambda'}{\lambda} - \frac{1}{r} = 0 \ . \tag{7}$$

This equation has the solution

$$\lambda(r) = \frac{\rho^2 \phi'}{r} \ , \tag{8}$$

where ρ is an integration constant with units of length.

Inserting (8) into (6) reduces the system to only one independent equation. Using either the rr, or $\theta\theta$ equation of (6) and multiplying by $r\phi^4$ gives

$$r\phi^3\phi'' + \phi^3\phi' + 3r\phi^2(\phi')^2 = \frac{\rho^2\phi^4\phi'}{2M^4}[K(r) - \Lambda] \ . \tag{9}$$

In [9] the source functions $F(r)$ and $K(r)$ outside the core $r > \epsilon$ were taken to have the form (f is some constant)

$$F(r > \epsilon) = K(r > \epsilon) = \frac{f}{\phi^2} \ , \tag{10}$$

The following $\phi(r)$ and $\lambda(r)$ are then solutions of the Einstein equations

$$\phi(r) = a \tanh\left[\frac{b}{2}\ln\left(\frac{r}{c}\right)\right] = a\frac{r^b - c^b}{r^b + c^b} \ , \qquad \lambda(r) = \frac{2bc^b\rho^2 r^{b-2}}{(r^b + c^b)^2} \tag{11}$$

where

$$a = \sqrt{\frac{5f}{3\Lambda}} \ , \qquad b = \frac{a\Lambda\rho^2}{5M^4} \ , \qquad c^b = \epsilon^b\frac{a-1}{a+1} \tag{12}$$

are integration constants. The solution (11) is an increasing function from the brane to some finite value at infinity

$$\phi(\infty) = a = \sqrt{\frac{5f}{3\Lambda}} > 1 \ . \tag{13}$$

The factor $1/\phi^2(r)$ has δ-like behavior outside the core and the source functions (10) decrease as required.

In [9] it was shown that the solution (11) provides a universal, gravitational trapping for all kinds of matter fields. However, in this model we did not specify source functions on the brane $0 \leq r \leq \epsilon$ and there were a large number of free parameters. Now we want to choose the source functions $F(r)$ and $K(r)$ everywhere, so that the solution ϕ will localize all kind of physical fields and be a regular function in the full 6-dimensional space-time.

We require for ϕ the following boundary conditions near the origin $r = 0$

$$\phi(r \to 0) \approx 1 + dr^2 \ , \quad \phi'(r \to 0) \approx 2dr \ , \tag{14}$$

where d is some constant. At infinity we want $\phi(r)$ to behave as

$$\phi(r \to \infty) \to a \ , \qquad \phi'(r \to \infty) \to 0 \ , \tag{15}$$

where $a > 1$ is some constant. Since the function ϕ' is proportional to the metric of the extra 2-space, the boundary conditions (15) imply that at infinity $\lambda \to 0$ and the effective geometry is 4-dimensional.

The source functions $F(r)$ and $K(r)$, which satisfy restriction (5) and give a desirable solution were found recently in the paper [12]

$$F(r) = \frac{f_1}{2\phi^2} + \frac{3f_2}{4\phi} \ , \qquad K(r) = \frac{f_1}{\phi^2} + \frac{f_2}{\phi} \ , \tag{16}$$

where f_1, f_2 are constants.

Substituting (16) into (9), taking its first integral and setting the integration constant to zero yields [8, 12]

$$r\phi' = \frac{\rho^2\Lambda}{10M^4}\left(\frac{5f_1}{3\Lambda} + \frac{5f_2}{4\Lambda}\phi - \phi^2\right) \ . \tag{17}$$

By introducing the parameters A and a such that

$$\frac{\rho^2\Lambda}{10M^4} = A \ , \qquad f_1 = -\frac{3\Lambda}{5}a \ , \qquad f_2 = \frac{4\Lambda}{5}(a+1) \ , \tag{18}$$

equation (17) becomes

$$r\phi' = A[-a + (a+1)\phi - \phi^2] \ . \tag{19}$$

Equation (19) is easy to integrate [12]

$$\phi = \frac{c^b + ar^b}{c^b + r^b} \ , \tag{20}$$

where $b = A(a-1)$ and c are integration constants. From the boundary conditions (14) it follows

$$b = A(a-1) = 2 \ . \tag{21}$$

The width of the brane ϵ corresponds to the inflection point of the function ϕ. Thus the condition $\phi''(r = \epsilon) = 0$ fixes c in (20) as $c^2 = 3\epsilon^2$. Finally the solution ϕ corresponding to a non-singular transverse gravitational potential for the brane has the form

$$\phi = \frac{3\epsilon^2 + ar^2}{3\epsilon^2 + r^2} \ . \tag{22}$$

From the condition that we have a 6-dimensional Minkowski metric on the brane, $\lambda(r = 0) = 1$, we can fix also the integration constant in (8)

$$\rho^2 = \frac{3\epsilon^2}{2(a-1)} \ . \tag{23}$$

Then using (21) the brane width can be expressed in terms of the bulk cosmological constant and fundamental scale

$$\epsilon^2 = \frac{40M^4}{3\Lambda} \ . \tag{24}$$

The metric tensor of the transverse space (8) is

$$\lambda = \frac{9\epsilon^4}{(3\epsilon^2 + r^2)^2} \ . \tag{25}$$

Note that it does not depend on a. In [10] it was shown that the solution given by (22) and (25) satisfied the condition of classical stability that the total momentum of the brane matter in the direction of the extra dimensions be zero [13]. Even though the coordinate r runs from 0 to ∞ the proper distance along r is finite

$$s = \int ds = \int_0^\infty \sqrt{\lambda(r)}dr = \int_0^\infty \frac{3\epsilon^2}{3\epsilon^2 + r^2}dr = \frac{\pi\sqrt{3}\epsilon}{2} \tag{26}$$

Thus this solution is more closely related to the two brane model with a finite distance between the branes (first reference in [3]) rather than the one brane model with an infinite extra dimension ([2] and second reference in [3]).

Using (20), (25) and the relationship (8) to integrate the gravitational part of the action integral (1) over the extra coordinates gives [9, 10]

$$S_g = \frac{M^4}{2}\int dx^6\sqrt{-^6g}\ ^6R = \frac{M^4}{2}\epsilon^2\pi(a^2 + a + 1)\int dx^4\sqrt{-\eta}\ R \ , \tag{27}$$

R and η are the scalar curvature and determinant, in four dimensions.

The formula for the effective Planck scale in our model, which is two times the numerical factor in front of the last integral in (27)

$$m_{Pl}^2 = M^4 \pi \epsilon^2 (a^2 + a + 1) \ , \tag{28}$$

is similar to those from the "large" extra dimensions model [1]. The differences are, the presence of the value of gravitational potential at infinity, a, in (28), and that the radius of the extra dimensions is replaced by the brane width ϵ, which, as seen from (24), is expressed by the ratio of the fundamental scale M and the cosmological constant Λ.

The normalization condition for a physical field, that its action integral over the extra coordinates r, θ converges, is also the condition for its localization. As was shown in [9] Newtonian gravity is localized on the brane, since the action integral for gravity, (27), is convergent over the extra space. However, the wave-functions of a localized matter field can be spread out from the brane more widely then the brane width ϵ. In order not to have contradictions with experimental facts, such as charge conservation [7], the parameters of the model must be chosen in a proper way.

When wave-functions of matter fields in six dimensions are peaked near the brane in the transverse dimensions their wave-functions on the brane can be factorized as

$$\Xi(x^A) = \frac{\xi(x^\nu)}{\kappa} \ , \tag{29}$$

where the parameter κ is the value of the constant zero mode with the dimension of length. These parameters can be found from the normalization condition for zero modes

$$\int_0^{2\pi} d\theta \int_0^\infty dr \sqrt{-^6 g} \frac{1}{\kappa^2} = \sqrt{-\eta} \ , \tag{30}$$

which also guarantees the validity of the equivalence principle for different kinds of particles.

Let us consider the situation with the localization of particular matter fields. If we assume that the zero mode of a spin-0 field, Φ, in six dimensions is independent of the extra coordinates its action takes the form [9, 10]

$$S_\Phi = \int d^6x \sqrt{-^6 g}\ {}^6L_\Phi(x^A) = \frac{\epsilon^2 \pi (a^2 + a + 1)}{\kappa_\Phi^2} \int d^4x \sqrt{-\eta} L_\Phi(x^\nu) \ , \tag{31}$$

where $L_\Phi(x^\nu)$ is the ordinary 4-dimensional Lagrangian of the spin-0 field and κ_Φ is value of the constant zero mode. The integral over r, θ in (31) is finite and the spin-0 field is localized on the brane.

The action for a vector field in the case of constant extra components ($A_i = const$) also reduces to the 4-dimensional Yang-Mills action multiplied an integral over the extra coordinates [9, 10]

$$S_A = \int d^6x\sqrt{-^6g}\ {}^6L_A(x^B) = \frac{3\epsilon^2\pi}{\kappa_A^2}\int d^4x\sqrt{-\eta}L_A(x^\nu)\ , \tag{32}$$

where κ_A is the value of the zero mode of the vector field. The extra integral in (32) is also finite and the gauge field is localized on the brane.

The factorization of the zero mode of a 6-dimensional spinor field in the ansatz (3) is different from the definition (29), having instead the form [9]

$$\Psi(x^A) = \frac{\psi(x^\nu)}{\kappa_\Psi\phi^2\left(r\phi'\right)^{1/4}}\ , \tag{33}$$

where κ_Ψ is a normalization constant. Integrating the 6-dimensional action of fermions over the extra coordinates, using (22), yields [9, 10]

$$S_\Psi = \int d^6x\sqrt{-^6g}\ {}^6L_\Psi(x^A) = \frac{3\pi^2\epsilon^2}{\kappa_\Psi^2\sqrt{2a(a-1)}}\int d^4x\sqrt{-\eta}L_\Psi(x^\nu)\ , \tag{34}$$

where $L_\Psi(x^\nu)$ is the 4-dimensional Dirac Lagrangian. The extra $1/\phi$ dependence in the second integral of (34) comes from the tetrad functions with upper index in the definition of the Dirac gamma matrices for the ansatz (3). The integral in (34) over r and θ is finite and Dirac fermions are localized.

Equating the coefficients of action integrals (31), (32) and (34) to 1, to satisfy the normalization condition (30), and guarantee the equivalence principle for gravity, we find the zero mode values for spin 0, 1 and 1/2 fields

$$\kappa_\Phi^2 = \pi\epsilon^2(a^2+a+1)\ , \qquad \kappa_A^2 = 3\pi\epsilon^2\ , \qquad \kappa_\Psi^2 = \frac{3\pi^2\epsilon^2}{\sqrt{2a(a-1)}}\ , \tag{35}$$

which are used to parameterize the 4-dimensional fields in the Lagrangians.

Within our model we now want to find the the positions where the zero modes are localized as well as the localization radii of the different fields.

From (31) the zero mode wave-function of the scalar field in flat space can be defined as

$$\Phi_0(r) = \sqrt{\frac{2\pi r\phi^2\lambda}{\kappa_\Phi^2}} = \frac{\sqrt{2\pi}3\epsilon^2}{\kappa_\Phi}\sqrt{r}\frac{(3\epsilon^2 + ar^2)}{(3\epsilon^2 + r^2)^2} \ , \tag{36}$$

where κ_Φ has the value (35). Function (36) is zero at infinity ($r \to \infty$) and on the brane ($r = 0$) and has a maximal value at some localization distance, d_Φ, between the brane ($r = 0$) and infinity ($r = \infty$). This localization distance, d_Φ, and the localization radius r_Φ, can be found by equating the first and second derivatives (to find the maximum and inflection point of the function) of (36) to zero respectively. For the localization distance this yields $d_\Phi^2 = \left(\epsilon^2(5a - 7 + \sqrt{49 - 58a + 25a^2})\right)/2a$. From this formula we see that since $a > 1$ the maximum of the wave-function of scalar fields (36) is located outside the brane $d_\Phi > \epsilon$. Setting the second derivative of (36) to zero gives $5ar^6 + (63 - 66a)\epsilon^2 r^4 - (102 - 45a)\epsilon^4 r^2 - 9\epsilon^6 = 0$. This is an effectively cubic equation for r^2, which has one real and two complex solutions. The radius, r_Φ, of the zero mode scalar wave-function is given by the real solution to this cubic equation. The lengthy expressions of the real solution can be obtained using a symbolic mathematics program such as *Mathematica*.

From (32) the wave-function of the vector field zero mode takes the form

$$A_0(r) = \sqrt{\frac{2\pi r\lambda}{\kappa_A^2}} = \frac{\sqrt{2\pi}3\epsilon^2}{\kappa_A}\frac{\sqrt{r}}{3\epsilon^2 + r^2} \ . \tag{37}$$

This function is also zero on the brane and at the infinity, and has a maximal value at some distance, d_A, in between. Again setting the first and second derivatives of (37) to zero we find $d_A = \epsilon$, and $r_A = \sqrt{\left(\epsilon^2(9 + 4\sqrt{6})\right)/5} \approx 1.9\epsilon$. So the peaks of the vector wave-functions are located exactly at the edge of the brane, $r = \epsilon$ and the radius of localization is approximately 2ϵ.

For the fermionic zero modes from (34) we have

$$\psi_0(r) = \sqrt{\frac{2\pi\rho^2}{\kappa_\Psi^2}\left(\frac{\phi'}{r\phi^2}\right)^{1/2}} = \left[\frac{54\pi^2\epsilon^6}{\kappa_\Psi^4(a-1)}\right]^{1/4}\frac{1}{\sqrt{3\epsilon^2 + ar^2}} \ . \tag{38}$$

This function is zero at infinity, but unlike the wave-functions of the scalar and vector zero modes, the peak of the fermion wave-function coincides with brane location, $r = 0$. From the inflection point of the function (38), which is found by equating the second derivative of (38) to zero, we obtain the localization radius for fermions $r_\Psi = \epsilon\sqrt{\frac{3}{2a}}$. Since $a > 1$ this formula indicates that the fermionic wave-functions are sharply peaked on the brane.

To summarize, we have shown that for a realistic form of the brane stress-energy, there exists a static, non-singular solution of the 6-dimensional Einstein equations, which provides a gravitational trapping of 4-dimensional gravity and matter fields on the brane. An essential feature of the model is that different kinds of matter fields have different localization distances from the brane. This property is in principle experimentally testable.

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