Chains of Monopoles and Closed Vortices

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Abstract

We consider new static axially symmetric solutions of SU(2) Yang-Mills-Higgs theory which represent chains of m monopoles and antimonopoles in static equilibrium. For monopoles and antimonopoles of higher charges totally different configurations arise, where the Higgs field vanishes on one or more rings, centered around the symmetry axis.

Introduction

Magnetic monopoles arise in spontaneously brocken gauge theories, when a semi-simple gauge group is brocken down to a subgroup containing an explicit $U(1)$ factor. The simplest monopole solution is the spherically symmetric 't Hooft-Polyakov monopole of SU(2) Yang-Mills-Higgs (YMH) theory [1, 2], which has unit topological charge. SU(2) multimonopoles carrying higher topological charge cannot be spherically symmetric [3]. They possess at most axial symmetry [4, 5, 6, 7, 8], or no rotational symmetry at all [9, 10].

In the Bogomol'nyi-Prasad-Sommerfield (BPS) limit of vanishing Higgs potential, the monopole and multimonopole solutions satisfy the first order Bogomol'nyi equations [11]. The spherically symmetric BPS monopole solution [12] and the axially symmetric BPS multimonopole solutions are known analytically [5, 6, 7]. For these solutions all nodes of the Higgs field are superimposed at a single point. BPS multimonopole solutions with only discrete symmetries have recently been constructed numerically [10]. In these solutions the nodes of the Higgs field can be located at several isolated points.

The energy of the BPS solutions satisfies exactly the lower energy bound given by the topological charge. Moreover, since in the BPS limit the repulsive and attractive forces between monopoles exactly compensate, BPS monopoles experience no net interaction [13].

However, as shown by Taubes, each topological sector contains besides the BPS monopole solutions further smooth, finite energy solutions. These do not satisfy the Bogomol'nyi equations, however, but only the second order field equations [14, 15]. Consequently, the energy of these solutions exceeds the Bogomol'nyi bound. The simplest such solution resides in the topologically trivial sector and forms a saddlepoint of the energy functional [14]. It possesses axial symmetry, and the two nodes of its Higgs field are located symmetrically on the positive and negative z-axis. This solution corresponds to a monopole-antimonopole pair in static equilibrium [16, 17].

In this note we discuss new axially symmetric solutions, where the Higgs field vanishes either at $m > 2$ isolated points on the symmetry axis [18] (chains of single monopoles and antimonopoles in alternating order) or the chains of charge 2-monopoles and charge 2-antimonopoles [19, 20]. For monopoles of charge $n > 2$ we have encountered a new phenomenon [20, 21]. Here the Higgs field vanishes on rings centered around the symmetry axis, instead of vanishing only at isolated points on the symmetry axis. These configurations represent the vortex rings.

SU(2) Yang-Mills-Higgs action and Ansätze

The Lagrangian density of SU(2) Yang-Mills-Higgs theory is given by

$$-L = \frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{4}\mathrm{Tr}\left(D_\mu\Phi D^\mu\Phi\right) + \frac{\lambda}{8}\mathrm{Tr}\left[\left(\Phi^2-\eta^2\right)^2\right] \ , \tag{1}$$

with $su(2)$ field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$, gauge potential $A_\mu = A_\mu^a\tau^a/2$, and covariant derivative of the Higgs field $\Phi = \phi^a\tau^a$ in the adjoint representation $D_\mu\Phi = \partial_\mu\Phi + ie[A_\mu, \Phi]$. Here e denotes the gauge coupling constant, η the vacuum expectation value of the Higgs field and λ the scalar coupling.

The general set of field equations is derived from the Lagrangian by variation with respect to the gauge potential and the Higgs field,

$$D_\mu F^{\mu\nu} - \frac{1}{4}ie\left[\Phi, D^\nu\Phi\right] = 0, \quad D_\mu D^\mu\Phi - \lambda\left(\Phi^2-\eta^2\right)\Phi = 0 \ . \tag{2}$$

Static finite energy configurations of the theory are labeled by a topological charge Q,

$$Q = \frac{1}{2!4\pi} \int\limits_{S_2} \epsilon_{abc} \hat{\Phi}^a d\hat{\Phi}^b \wedge \hat{\Phi}^c = -\frac{i}{16\pi} \int\limits_{S_2} \mathrm{Tr}\ \left(\hat{\Phi}\ d\hat{\Phi} \wedge d\hat{\Phi} \right) \ , \tag{3}$$

where $\hat{\Phi}$ is the normalized Higgs field, $|\hat{\Phi}| = 1/2\,\mathrm{Tr}\,\hat{\Phi}^2 = \sum_a (\hat{\Phi}^a)^2 = 1$. The topological charge is thus the topological mapping index for the map from the 2-sphere of spatial infinity S_2^{phys} to the 2-sphere of internal space S_2^{int}, representing the vacuum manifold. Static finite energy configurations therefore fall into topological sectors.

The topological charge is associated with the conserved topological current $k_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu\hat{\Phi}^a\partial^\rho\hat{\Phi}^b\partial^\sigma\hat{\Phi}^c$, i.e., $Q = \int k_0 d^3r$.

In the BPS limit the energy of static field configurations takes the form

$$E = \int \frac{1}{4}\left(F_{ij}^a \mp \varepsilon_{ijk} D_k \Phi^a\right)^2 d^3r \pm \int \frac{1}{2}\varepsilon_{ijk} F_{ij}^a D_k \Phi^a d^3r \ . \tag{4}$$

For a configurations with topological charge Q it is bounded from below $E \geq \frac{4\pi\eta Q}{e}$; the monopole and multimonopole solutions satisfying the first order Bogomol'nyi equations

$$F_{ij} = \pm\frac{1}{2}\varepsilon_{ijk} D_k \Phi \tag{5}$$

precisely saturate the lower energy bound. We here consider solutions which, even in the limit of vanishing Higgs selfcoupling, $\lambda = 0$, do not saturate the Bogomol'nyi bound.

To obtain static axially symmetric solutions, we parametrize the gauge potential and the Higgs field by the Ansätze

$$\begin{aligned} A_\mu dx^\mu &= \left(\frac{K_1}{r}dr + (1-K_2)d\theta\right)\frac{\tau_\varphi^{(n)}}{2e} \\ &\qquad -n\sin\theta\left(K_3\frac{\tau_r^{(n,m)}}{2e} + (1-K_4)\frac{\tau_\theta^{(n,m)}}{2e}\right)d\varphi \ , \end{aligned} \tag{6}$$

$$\Phi = \Phi_1 \tau_r^{(n,m)} + \Phi_2 \tau_\theta^{(n,m)} \ . \tag{7}$$

where the $su(2)$ matrices $\tau_r^{(n,m)}$, $\tau_\theta^{(n,m)}$, and $\tau_\varphi^{(n)}$ are defined as products of the spatial unit vectors

$$\begin{aligned} \hat{e}_r^{(n,m)} &= (\sin(m\theta)\cos(n\varphi), \sin(m\theta)\sin(n\varphi), \cos(m\theta)) \ , \\ \hat{e}_\theta^{(n,m)} &= (\cos(m\theta)\cos(n\varphi), \cos(m\theta)\sin(n\varphi), -\sin(m\theta)) \ , \\ \hat{e}_\varphi^{(n)} &= (-\sin(n\varphi), \cos(n\varphi), 0) \ , \end{aligned} \tag{8}$$

with the Pauli matrices $\tau^a = (\tau_x, \tau_y, \tau_z)$, i.e.

$$\begin{aligned}
\tau_r^{(n,m)} &= \sin(m\theta)\tau_\rho^{(n)} + \cos(m\theta)\tau_z \ , \\
\tau_\theta^{(n,m)} &= \cos(m\theta)\tau_\rho^{(n)} - \sin(m\theta)\tau_z \ , \\
\tau_\varphi^{(n)} &= -\sin(n\varphi)\tau_x + \cos(n\varphi)\tau_y \ ,
\end{aligned}$$

with $\tau_\rho^{(n)} = \cos(n\varphi)\tau_x + \sin(n\varphi)\tau_y$. We refer to the integers n and m in (8),(6),and (7) as the θ winding number and the φ winding number, respectively. Indeed, as the unit vector (8) parametrized by the polar angle θ and azimuthal angle φ covers the sphere S_2 once, the fields given by the Ansatz (6) and (7) wind n and m times around the z-axis and the ρ-axis, respectively.

The functions $K_1 - K_4$ and Φ_1, Φ_2 depend on the coordinates r and θ only. Thus, this ansatz is axially symmetric since a spatial rotation around the z-axis can be compensated by a gauge transformation

$$U = \exp\{i\Gamma(r,\theta)\tau_\varphi^{(n)}/2\} \ , \tag{9}$$

which leaves the ansatz form-invariant [22].

These Ansätze [22, 20] generalize both the Ansätze employed in [16, 17, 19, 18] for the monopole-antimonopole pairs and chains, as well as the axially symmetric multimonopole Ansätze [4, 8].

To obtain regular solution with finite energy density and appropriate asymptotic behavour, we have to impose certain boundary conditions which we discussed in [20, 21]. Here we briefly present the general properties of the solutions.

Electromagnetic properties

A gauge-invariant definition of the electromagnetic field strength tensor is given by the 't Hooft tensor [1]

$$\mathcal{F}_{\mu\nu} = \mathrm{Tr}\left\{\hat{\Phi}F_{\mu\nu} - \frac{i}{2e}\hat{\Phi}D_\mu\hat{\Phi}D_\nu\hat{\Phi}\right\} = \hat{\Phi}^a F^a_{\mu\nu} + \frac{1}{e}\epsilon_{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c \ . \tag{10}$$

The 't Hooft tensor then yields the electric current j^ν_{el}

$$\partial_\mu\mathcal{F}^{\mu\nu} = 4\pi j^\nu_{\mathrm{el}} \ , \tag{11}$$

and the magnetic current j^ν_{mag}

$$\partial_\mu{}^*\mathcal{F}^{\mu\nu} = 4\pi j^\nu_{\mathrm{mag}} \ . \tag{12}$$

Since the magnetic current is proportional to the topological current, $ej^{\nu}_{\rm mag} = k^{\nu}$, the magnetic charge g is given by

$$\begin{aligned} g &= \frac{Q}{e} = \int \frac{k_0}{e} d^3 r \\ &= \frac{1}{4\pi} \int \vec{\nabla} \cdot \vec{\mathcal{B}} d^3 r \ , \end{aligned} \tag{13}$$

with magnetic field $\mathcal{B}_i = \frac{1}{2}\epsilon_{ijk}\mathcal{F}^{jk}$. Alternatively, the magnetic charge can also be obtained from

$$g = \frac{1}{4\pi\eta} \int \frac{1}{2} {\rm Tr}\ (F_{ij} D_k \Phi)\, \varepsilon_{ijk} d^3 r \ . \tag{14}$$

Evaluation of the 't Hooft tensor Eq. (10) with the above Ansatz yields $\mathcal{F}_{\theta\varphi} = \partial_\theta \mathcal{A}_\varphi \ , \quad \mathcal{F}_{\varphi r} = -\partial_r \mathcal{A}_\varphi \ , \quad \mathcal{F}_{r\theta} = 0$, with

$$\begin{aligned} \mathcal{A}_\varphi = \frac{n}{e} \Big[-\hat{\Phi}_1 \left[K_3 \sin\theta + \cos(m\theta) \right] \\ + \hat{\Phi}_2 \left[(K_4 - 1)\sin\theta + \sin(m\theta) \right] \Big] \ , \end{aligned} \tag{15}$$

and $\hat{\Phi}_1 = \Phi_1 / \sqrt{\Phi_1^2 + \Phi_2^2}$, $\hat{\Phi}_2 = \Phi_2 / \sqrt{\Phi_1^2 + \Phi_2^2}$.

Thus, the contour lines of the vector potential component $\mathcal{A}_\varphi$, correspond to the field lines of the magnetic field $\vec{\mathcal{B}}$.

Evaluation of the magnetic charge then yields

$$\begin{aligned} g = \frac{1}{4\pi} \int_{S_2(\infty)} \mathcal{F}_{\theta\varphi} d\theta d\varphi = -\frac{n}{2e} \int \partial_\theta \left(K_3|_{r=\infty} \sin\theta + \cos(m\theta) \right) d\theta \\ = \frac{n}{2e} \left[1 - (-1)^m \right] \ , \end{aligned} \tag{16}$$

that is the magnetic charge vanishes for even θ winding number m, and it is proportional φ winding number n for odd θ winding number m.

The magnetic dipole moment can be obtained from the magnetic charge density and the electric current density [21, 23],

$$\vec{\mu} = \int \left(\vec{r} \frac{k_0}{e} + \frac{1}{2} \vec{r} \times \vec{j}_{\rm el} \right) d^3 r \ , \tag{17}$$

where k_0/e and $\vec{j}_{\rm el}$ are the magnetic charge density (12) and the electric current density (11), respectively. Thus, the physical picture of the source of the dipole moment is that it originates both from a distribution of magnetic charges and electric currents. Because of axial symmetry of the configurations, $\vec{\mu} = \mu\vec{e}_z$. The contribution of the electric current density to the magnetic moment is obtained from

$$\begin{aligned}\mu_{\rm current} &= \frac{1}{2}\int j_\varphi r^2 \sin\theta dr d\theta d\varphi \\ &= \frac{1}{4}\int dr d\theta \left[r^2 \sin\theta \partial_r^2 \mathcal{A}_\varphi + \sin^2\theta \partial_\theta \frac{1}{\sin\theta}\partial_\theta \mathcal{A}_\varphi\right] \qquad (18)\end{aligned}$$

Monopole-Antimonopole Chains and Vortex Rings

Monopole-antimonopole chains

Here we briefly desctibe the properties of the numerical soultions [18, 20, 21]. Note that all of them may be characterized by a global 2d Poincare index $i_\infty = \frac{m}{2}[1-(-1)^n]$ of the Higgs field which is complementary to the 3-dimensional invariant (16) [24]

The solution of the first type are m-chains constructed in [18, 20]. They are characterized by θ winding number $m > 1$ and φ winding number $n = 1, 2$.

Let us consider $n = 1$ configurations first. These m-chains possess m nodes of the Higgs field on the z-axis. Due to reflection symmetry, each node on the negative z-axis corresponds to a node on the positive z-axis. The nodes of the Higgs field $x_0(k)$ are associated with the location of the monopoles and antimonopoles. For an odd m ($m = 2k + 1$) the Higgs field possesses k nodes on the positive z-axis and one node at the origin. The node at the origin corresponds to a monopole if k is even and to an antimonopole if k is odd. For even m ($m = 2k$) there is no node of the Higgs field at the origin.

The $m = 1$ solution is the spherically symmetric 't Hooft-Polyakov monopole. The $m = 3$ (M-A-M) and $m = 5$ (M-A-M-A-M) chains represent deformations of the sector with unit topological charge. The $m = 2$ chain is identical to the monopole-antimonopole (M-A) pair discussed in [16, 17]. The M-A pair as well as the $m = 4$ (M-A-M-A) and $m = 6$ (M-A-M-A-M-A) configurations are deformations of the vacuum sector with zero topological charge. In Fig. 1 we present the dimensionless energy density and nodes of the Higgs field for the $n = 1$ solutions with θ winding number $m = 1, \ldots, 6$. The energy density of the m-chain possesses m maxima on the z-axis, and decreases with increasing ρ. The locations of the maxima are close to the nodes of the Higgs field. For a given m the maxima are of similar magnitude, but their height decreases with increasing m. Increasing of λ makes these maxima sharper and decreases the distance between the locations of the monopoles. At the

Figure 1: The rescaled energy density $E(\rho, z)$ is shown for the monopole-antimonopole chains with winding numbers $n = 1, m = 1, \ldots 6$ at $\lambda = 0$.

same time, the modulus of the Higgs field tends faster and further towards its vacuum expectation value inbetween the locations of the monopoles. We observe that for a given λ the distances between the corresponding nodes increase with increasing m.

We observe that the energy $E^{(m)}$ of an m-chain is always smaller than the energy of m single monopoles or antimonopoles (with infinite separation between them), i. e. $E^{(m)} < E_\infty = 4\pi\eta m$. On the other hand $E^{(m)}$ exceeds the minimal energy bound given by the Bogolmol'ny limit $E_{\rm min}/4\pi\eta = 0$ for even m, and $E_{\rm min}/4\pi\eta = 1$ for odd m. We observe an (almost) linear dependence of the energy $E^{(m)}$ on m. This can be modelled by taking into account only the energy of m single (infinitely separated) monopoles and the next-neighbour interaction between monopoles and antimonopoles on the chain [18, 21].

If $\lambda = 0$ we interpret the m-chains as equilibrium states of m non-BPS monopoles and antimonopoles. As shown long ago [13], the force between BPS monopoles is given by twice the Coloumb force when the charges are unequal, and vanishes when the charges are equal.. Thus, monopoles and antimonopoles can only be in static equilibrium, if they are close enough to experience a repulsive force that counteracts the attraction.

Indeed, let us consider the magnetic dipole. Numerical solution shows that the separation between monopole and antimonopole is not too large, of order of 4 v.e.v of the scalar field. Thus, both photon and scalar particles remain massive and together with the vector bosons $A_\mu^\pm$ they all contribute to the short-range Yukawa-type interactions.

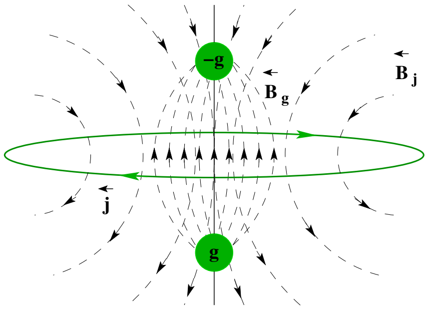

Figure 2: Magnetic dipole: the abelian magnetic fields of the charges and current loop

At the first glance there is no place for a repulsive force: both electromagnetic and scalar interactions between the monopoles are attractive whereas the net effect of interaction mediated by the vector bosons is zero. However, the structure of the solution shows that, according to the equations (11) (12), the $U(1)$ magnetic field, which is defined by the ‘t Hooft tensor (10), is generated both by the monopoles and by the electric current ring (cf Fig. 2).

There is the magnetic field of the current loop which stabilizes the configuration. In the BPS limit such a configuration is proved to be an equilibrium state [14]. It looks plausible that for a very heavy scalar field such a magnetic dipole corresponds to the local minimum of the energy functional.

Note that these configurations are essentially non-BPS solutions. To see this in another way let us consider the limit $\lambda = 0$. Then the mass of these “non-Bogomol’nyi BPS monopoles” can be written in the form (4). The second term there is proportional to the topological charge and vanishes when n is even while the first term is just the integral of the square of the Bogomol’nyi equations. Thus, for even m the energy is a measure for the deviation of the solution from selfduality.

Let us now consider chains consisting of multimonopoles with winding number $n = 2$. Identifying the locations of the Higgs zeros on the symmetry axis with the locations of the monopoles and antimonopoles, we observe that when each pole carries charge $n = 2$, the zeros form pairs, where the distance between the monopole and the antimonopole of a pair is less than the distance to the neighboring monopole or antimonopole, belonging to the next pair.

We observe furthermore, that the equilibrium distance of the monopole-antimonopole pair composed of $n = 2$ multimonopoles is smaller than the equilibrium distance of the monopole-antimonopole pair composed of single monopoles. Thus the higher attraction between the poles of a pair with charge $n = 2$ is balanced by the repulsion only at a smaller equilibrium distance.

The difference from the chains composed of pairs of monopole-antimonopole with winding number $n = 1$ we considered above, is that for $n = 2$ chains maxima of the energy density are no longer coincide with (double) zeros of the Higgs field. One may undestand the reason by consideration of the energy associated with the electromagnetic current which has to be much stronger to balance the attraction between two monopoles of double charge.

Closed vortices

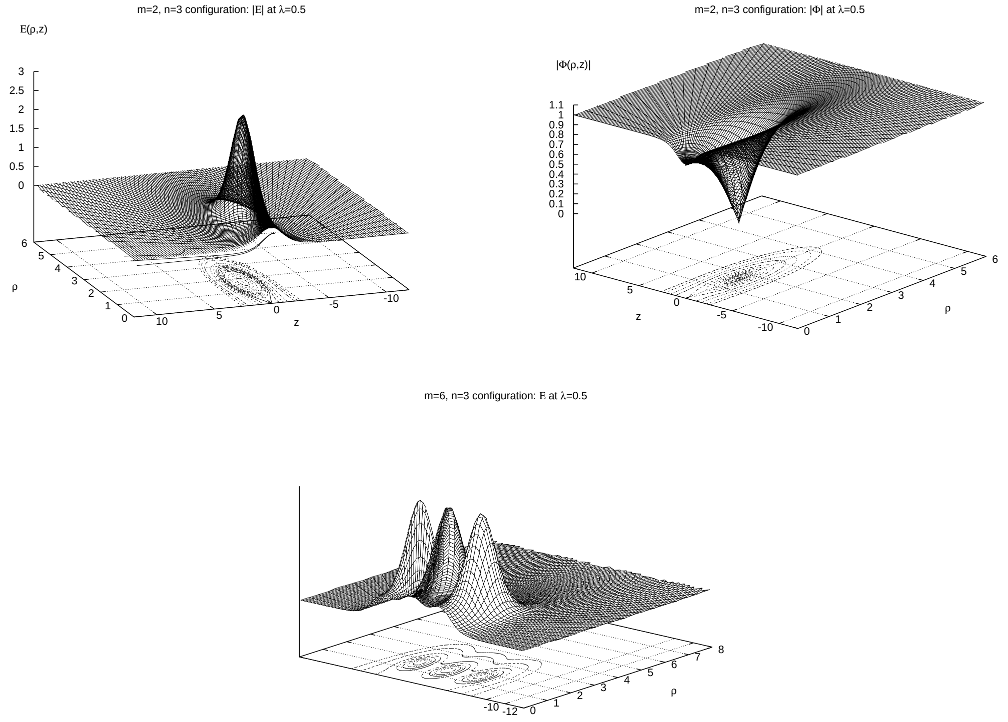

Figure 3: The rescaled energy density $E(\rho, z)$ and the modulus of the Higgs field $|\Phi(\rho, z)|$ are shown for the circular vortex solutions $m = 2, n = 3$ (single vortex), $m = 4, n = 3$ (double vortex) and $m = 6, n = 3$ (triple vortex) at $\lambda = 0.5$.

The situation changes dramatically if φ winding number $n > 2$. The solution of the second type, which appears in that case are not multimonopole chains but sustems of vortex rings, in the BPS limit either bounded with a single n-monopole placed at the origin, or without it [20]. One could expect that when charge of poles is increasing further beyond $n = 2$ the chain solutions consisting of multimonopoles with winding number n should exist, the monopoles and antimonopoles of the pairs should approach each other further, settling at a still smaller equilibrium distance.

Constructing solutions with charge $n = 3$ in the BPS limit ($\lambda = 0$) however, we do not find chains at all. Now there is no longer sufficient repulsion to balance the strong attraction between the 3-monopoles and 3-antimonopoles. Instead of chains, we now observe solutions with vortex rings, where the Higgs field vanishes on closed rings around the symmetry axis with increasing radius for increasing n. The physical 3-monopole-3-antimonopole solution in the BPS limit then has a single ring of zeros of the Higgs field and no point zeros.

Considering the magnetic moment of the $m = 2$ solutions, we observe, that it is proportional to n. The pair of poles on the z-axis for $n = 2$ clearly gives rise to the magnetic dipole moment of a physical dipole. The ring of zeros also corresponds to a magnetic dipole field, which however looks like the field of a ring of mathematical dipoles. This corresponds to the simple picture that the positive and negative charges have merged but not annihilated, and then spread out on a ring. Thus, we can identify the $m = 2$, $n = 3$ solution with a closed vortex configuration.

While the dipole moment of monopole-antimonopole chains with an equal number of monopoles and antimonopoles has its origin in the magnetic charges of the configuration [17, 18, 21], the dipole moment of the closed vortices is associated with loops of electric currents [21].

Other solutions with even θ winding number reside in the vacuum sector as well.

$n > 2$ and odd m: Vortices bounded with monopole

The solutions with odd θ winding number have an isolated zero of the Higgs field at the origin, thus their reside in the topological sector with charge n. For $m = 4k + 1$ the situation is somewhat similar to the above. Here a single n-monopole remains at the origin, whereas all other zeros form $2k$ rings carrying dipole strength. Since, however, a dipole on the positive axis and its respective counterpart on the negative axis have opposite orientation, their contributions cancel in the total magnetic moment. Thus the magnetic moment remains zero, as it must, because of the symmetry of the ansatz [18, 21]. Non-zero scalar coupling does not change the situation but the picks of the energy density are getting sharper.

For $m = 4k - 1$, on the other hand, the situation is more complicated. Let us consider the simplest case $m = 3$ in the limit $\lambda = 0$. We may consider unphysical configurations with winding number n continuously vary between $n = 2$ chain solution and $n = 3$ configuration. The difference from the case of an even m is that in the initial state there are 3 poles on the z-axis, which cannot form pairs, such that all zeros belong to a pair, symmetrically located around the origin, thus the dipole moment of that configuration is zero. For $m = 3$, we observe in the BPS limit, that two vortices appear in the charge $n = 3$ solution, emerging from the upper and lower unpaired zero, respectively, carrying opposite dipole strength. Increasing of scalar coupling decreases the radius of the vortices as well the distance between then. However both rings remain individual.

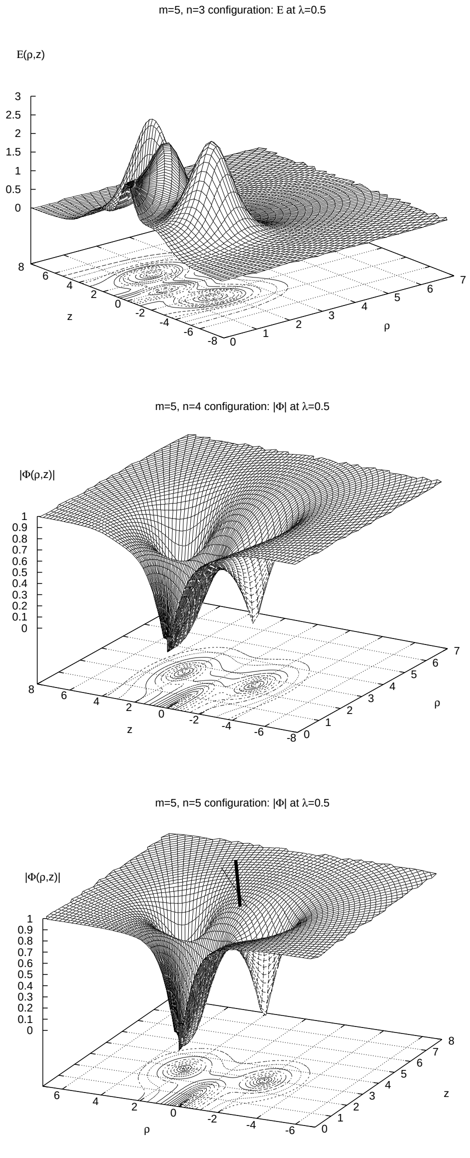

Figure 4: The rescaled energy density $E(\rho, z)$ and the modulus of the Higgs field $|\Phi(\rho, z)|$ are shown for the monopole-double vortex solution with winding numbers $m = 5, n = 3, 4, 5$ at $\lambda = 0.5$

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