From the Laurent-series Solutions to Elliptic Solutions of Nonintegrable systems

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Abstract

The Painlevé test is very useful to construct not only the Laurent-series solutions, but also the elliptic and trigonometric ones. To find the elliptic solutions one can transform a nonlinear polynomial differential equation in a nonlinear algebraic system in parameters of the Laurent-series solutions. This procedure can be automatized. The Painlevé test can also assist to solve the inverse problem: to find the form of a polynomial potential, which corresponds to the required type of solutions.

1   INTRODUCTION

The investigations of the exact special solutions of nonintegrable systems play an important role in the study of nonlinear physical phenomena. When some mechanic or field theory problem is studied, time is assumed to be real, whereas the integrability of motion equations is connected with the behavior of their solutions as functions of a complex coordinate. Consideration of the motion equation on complex (time) plane can help to determine type of possible real solutions. The analysis of solutions in the neighborhood of their singular points (the Painlevé test) is very useful to construct the elliptic and

trigonometric solutions. To do this one has to solve only algebraic equations, the algorithm [1] can be automatized due to computer algebra systems [2].

The Painlevé analysis assists also to solve the inverse problem: to define a polynomial potential, corresponding to the required type of solutions. In this paper we consider a few examples, which show how the local analysis can be used. In the next section we formulate the Painvelé property. In the third section we consider the five-dimensional gravitational model with a scalar field and seek the correspondence between the scalar field potential and type of this field. In the fourth section we compare the method of construction of trigonometric and elliptic solutions, based on the Painlevé analysis [1], with traditional ones.

2   THE PAINLEVÉ PROPERTY

Let us formulate the Painlevé property for ordinary differential equations (ODE's). Solutions of a system of ODE's are regarded as analytic functions, maybe with isolated singular points [3]. A singular point of a solution is said *critical* (as opposed to *noncritical*) if the solution is multivalued (single-valued) in its neighborhood and *movable* if its location depends on initial conditions. The *general solution* of an ODE of order N is the set of all solutions mentioned in the existence theorem of Cauchy, i.e. determined by the initial values. It depends on N arbitrary independent constants. A *special solution* is any solution obtained from the general solution by giving values to the arbitrary constants. A *singular solution* is any solution which is not special, i.e. which does not belong to the general solution. A system of ODE's has **the Painlevé property** if its general solution has no movable critical singular point [3, 4].

There exist two distinctions between the structure of solutions of linear differential equations and nonlinear ones. Linear ODE's have no singular solution and their general solutions have no movable singularity.

Investigations of many dynamical systems show that a system is completely integrable for such values of parameters, at which it has the Painlevé property. At the same time the integrability of an arbitrary system with the Painlevé property has yet to be proved. There is not an algorithm for construction of the additional integral by the Painlevé analysis. There exist many examples of integrable systems without the Painlevé property.

The Painlevé test is any algorithm, which checks some necessary con-

ditions for a differential equation to have the Painlevé property. The original algorithm, developed by P. Painlevé and used by him to find all the second order ODE's with Painlevé property, is known as the α-method. The method of S.V. Kovalevskaya [5] is not as general as the α–method, but much more simple. The remarkable property of this test is that it can be checked in a finite number of steps. This test can only detect the occurrence of logarithmic and algebraic branch points. To date there is no general finite algorithmic method to detect the occurrence of essential singularities[1]. In 1980, developing the Kovalevskaya method further, M.J. Ablowitz, A. Ramani and H. Segur [7] constructed a new algorithm of the Painlevé test for ODE's. This algorithm appears very useful to find solutions as a formal Laurent series. First of all, it allows to determine the dominant behavior of a solution in the neighborhood of the singular point t_0. If the solution tends to infinity as $(t - t_0)^\beta$, where β is a negative integer number, then substituting the Laurent series expansions one can transform nonlinear differential equations into a system of linear algebraic equations on coefficients of the Laurent series. All solutions of an autonomous system depend on the arbitrary parameter t_0, which characterizes the singular point location. If a single-valued solution depends on other parameters, then some coefficients of its Laurent series have to be arbitrary and the corresponding systems have to have zero determinants. The numbers of such systems (named *resonances* or *Kovalevskaya exponents*) can be determined due to the Painlevé test.

3 FIVE-DIMENSIONAL GRAVITATIONAL MODEL WITH A SCALAR FIELD

To show how the analysis of singular behavior of solutions can assist to find the form of potential, let us consider the model of gravity field interacting with a single scalar field in five-dimensional space-time [8, 9]. The action is

$$S = \int_M d^4x dr \sqrt{|det\, g_{\mu\nu}|} \left(-\frac{R}{4} + \frac{1}{2}(\partial\phi)^2 - V(\phi) \right), \qquad (1)$$

where M is the full five-dimensional space-time. The most general metric with four-dimensional Poincaré symmetry is

$$ds^2 = e^{2A(r)}(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2) - dr^2.$$

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[1]Different variants of the Painlevé test are compared in [6, R. Conte paper]

If the scalar field depends only on additional coordinate: $\phi = \phi(r)$, then the independent equations of motion are

$$H' = -\frac{2}{3}(\phi')^2, \tag{2}$$

$$H^2 = -\frac{1}{3}V(\phi) + \frac{1}{6}(\phi')^2, \tag{3}$$

where $H \equiv A' \equiv \frac{dA}{dr}$ and $\phi' \equiv \frac{d\phi}{dr}$.

Let us analyse the correspondence between type of the scalar field $\phi(r)$ and the potential $V(\phi)$. We assume that $V(\phi)$ is a polynomial of ϕ. If at singular point

$$\phi(r) \sim \frac{1}{r^m},$$

then from (2) we obtain

$$H' \sim \frac{1}{r^{2m+2}} \quad \Longrightarrow \quad H \sim \frac{1}{r^{2m+1}}.$$

From (3) it follows that

$$V(\phi) \simeq V\left(\frac{1}{r^m}\right) \sim \frac{1}{r^{4m+2}}. \tag{4}$$

It means that solutions with poles proportional $1/r$ can be obtained only if the power of the polynomial potential $V(\phi)$ is equal to six[2]. For example, let $\phi_1(r) = \tanh(r)$, for real r this function has no singular point, but equations (2) and (3) are autonomous ones, so if $\phi_1(r)$ is a solution, then $\phi_1(r - r_0)$, where r_0 is an arbitrary complex constant, is a solution too. Hence, the function $\phi_1(r)$ cannot be a solution for the standard ϕ^4 potential:

$$V(\phi) = (\phi^2 - 1)^2. \tag{5}$$

In [8] the explicit form of the sixth order polynomial potential $V(\phi)$, which corresponds to $\phi_1(r)$ has been found. If $\phi(r)$ is, for example,

$$\phi(r) = \sum_{k=1}^{N} \tanh(r - r_k),$$

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[2]Similar solutions can exist, surely, for nonpolynomial potentials as well.

where N is some natural number and r_k are some constants, then the explicit form of the corresponding potential is not known, but from the Painlevé analysis it follows that if $V(\phi)$ is a polynomial, then its degree has to be equal to 6. Analogously one can show that if solutions tend to infinity as $1/r^2$, then the power of $V(\phi)$ is equal to 5. If solutions tend to infinity as $1/r^k$, where k is a natural number greater than two, then $V(\phi)$ can not be a polynomial. In conclusion of this section we say a few words about explicit solutions and the correspondence between ϕ and $V(\phi)$. Following [8] we assume that $H(r)$ is a function of ϕ:

$$H(r) = -\frac{1}{3}W(\phi).$$

It is straightforward to verify that equations (2) and (3) are equivalent to

$$\frac{d\phi(r)}{dr} = \frac{1}{2}\frac{dW(\phi)}{d\phi}, \tag{6}$$

$$\left(\frac{dW(\phi)}{d\phi}\right)^2 - \frac{1}{3}W(\phi)^2 - V(\phi) = 0. \tag{7}$$

Unfortunately eq. (7) can not be solved analytically. For polynomial $V(\phi)$ it is possible, if possible, to find only special solutions: $W(\phi)$ in the polynomial form[3].

Contrary to a scalar field theory without gravitational field, there is not one to one correspondence between the form of the scalar field $\phi(r - r_0)$ and potential $V(\phi)$. The form of the scalar field is defined by $\frac{dW}{d\phi}$, so one can add a constant to $W(\phi)$ and obtain new $V(\phi)$ for the same $\phi(r)$. On the other hand, for given $V(\phi)$ we have not one-, but two-parameter set of functions $\phi(r)$.

4 SEARCH OF SPECIAL SOLUTIONS

4.1 The generalized Hénon-Heiles system

To analyse the methods of construction of special single-valued solutions let us consider the generalized Hénon–Heiles system with an additional non-

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[3]For example, $W(\phi)$ can not be a polynomial if $V(\phi) = (\phi^2 - 1)^2$ and we don't know a solution for this potential.

polynomial term, which is described by the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\left(x_t^2 + y_t^2 + \lambda_1 x^2 + \lambda_2 y^2\right) + x^2 y - \frac{C}{3}y^3 + \frac{\mu}{2x^2}$$

and the corresponding system of the motion equations:

$$\begin{cases} x_{tt} = -\lambda_1 x - 2xy + \dfrac{\mu}{x^3}, \\ y_{tt} = -\lambda_2 y - x^2 + Cy^2, \end{cases} \qquad (8)$$

where $x_{tt} \equiv \frac{d^2 x}{dt^2}$ and $y_{tt} \equiv \frac{d^2 y}{dt^2}$, λ_1, λ_2, μ and C are arbitrary numerical parameters. Note that if $\lambda_2 \neq 0$, then one can put $\lambda_2 = sign(\lambda_2)$ without loss of generality. If $C = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$ and $\mu = 0$, then (8) is the initial Hénon–Heiles system [10].

The function y, solution of system (8), satisfies the following fourth-order equation, which does not include μ:

$$y_{tttt} = (2C - 8)y_{tt}y - (4\lambda_1 + \lambda_2)y_{tt} + 2(C + 1)y_t^2 +$$
$$+ \frac{20C}{3}y^3 + (4C\lambda_1 - 6\lambda_2)y^2 - 4\lambda_1\lambda_2 y - 4H. \qquad (9)$$

We note that the energy of the system H is not an arbitrary parameter, but a function of initial data: y_0, y_{0t}, y_{0tt} and y_{0ttt}. The form of this function depends on μ:

$$H = \frac{y_{0t}^2 + y_0^2}{2} - \frac{C}{3}y_0^3 + \left(\frac{\lambda_1}{2} + y_0\right)(Cy_0^2 - \lambda_2 y_0 - y_{0tt}) +$$
$$+ \frac{(\lambda_2 y_{0t} + 2Cy_0 y_{0t} - y_{0ttt})^2 + \mu}{2(Cy_0^2 - \lambda_2 y_0 - y_{0tt})}. \qquad (1)$$

This formula is correct only if $x_0 \neq 0$. If $x_0 = 0$, what is possible only at $\mu = 0$, then we can not express x_{0t} through y_0, y_{0t}, y_{0tt} and y_{0ttt}, so H is not a function of the initial data. If $y_{0ttt} = 2Cy_0 y_{0t} - \lambda_2 y_{0t}$, then eq. (3) with an arbitrary H corresponds to system (8) with $\mu = 0$, in opposite case eq. (9) does not correspond to system (8).

The Painlevé test of eq. (9) gives the following dominant behaviors and resonance structures near the singular point t_0 [11]:

1. The function $y(t)$ tends to infinity as $b_{-2}(t - t_0)^{-2}$, where $b_{-2} = -3$ or $b_{-2} = \frac{6}{C}$.

2. For $b_{-2} = -3$ (*Case 1*) the values of resonances are

$$r = -1, \ 10, \ (5 \pm \sqrt{1 - 24(1 + C)})/2.$$

In *Case 2* ($b_{-2} = \frac{6}{C}$) $r = -1, \ 5, \ 5 \pm \sqrt{1 - 48/C}$.

The resonance $r = -1$ corresponds to an arbitrary parameter t_0 (the location of the singular point). Other values of r determine powers of t (their values are $r - 2$), at which new arbitrary parameters can appear as solutions of the linear systems with zero determinant. For integrability of system (8) all values of r have to be integer and all systems with zero determinants have to have solutions at any values of free parameters included in them. It is possible only in the three known integrable cases [12].

For the search for special solutions, it is interesting to consider such values of C, for which r are integer numbers either only in *Case 1* or only in *Case 2*. It has been shown in [12, 13] (for $\lambda_2 = 1$ and $\mu = 0$) and [11] (for arbitrary values of parameters) that single-valued three-parameter special solutions exist in two nonintegrable cases: $C = -16/5$ and $C = -4/3$ (λ_1 and λ_2 are arbitrary). When the resonance structure is known it is easy to write the computer algebra program, which finds the Laurent series solutions with an arbitrary accuracy (for example, we have found 65 coefficients).

4.2   Construction of Global Single-Valued Solutions

The classical method to find special analytic solutions for the generalized Hénon–Heiles system is the following:

1) Transform system (8) into eq. (9).

2) Assume that y satisfies some first order equation, substitute this equation in (9) and obtain a nonlinear algebraic system.

3) Solve the obtained system.

This method doesn't use the result of the Painleve test and the known Laurent series solutions. It may be difficult to automatize this alhorithm, because all its steps can be nontrivial.

The algorithm for finding special solutions for ODE's in the form of a finite expansion in powers of unknown function $\varphi(t - t_0)$ has been constructed in [14]. The function $\varphi(t - t_0)$ and coefficients have to satisfy some system of ODE, often more simple than an initial one. This method based on the Painlevé test, it does not transform differential equations to algebraic.

Differing from the above-mentioned methods, which do not use the Laurent series solutions of the initial nonintegrable system, the method [1] used them.

It has been proved by Fuchs [15] (see also [3]) that the necessary form of a polynomial autonomous first order ODE with the single-valued general solution is

$$\sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{jk}\, y^j y_t^k = 0, \qquad a_{0m} = 1, \tag{10}$$

in which m is a positive integer number and a_{jk} are constants.

The Briot and Bouquet theorem [16] proves that if the general solution of a polynomial autonomous first order ODE is single-valued, then this solution is either an elliptic function, or a rational function of $e^{\gamma x}$, γ being some constant, or a rational function of x. Note that the third case is a degeneracy of the second one, which in its turn is a degeneracy of the first one.

The proposed by R. Conte and M. Musette algorithm [1] is the following:

1) Choose a positive integer m and define the first order ODE (10), which contains unknown constants a_{jk}.

2) Compute coefficients of the Laurent series solutions for (8) or (9) with some fixed C. The number of coefficients has to be greater than the number of unknowns.

3) Substituting the obtained coefficients, transform eq. (8) into a linear and overdetermined system in a_{jk} with coefficients depending on arbitrary parameters.

4) Eliminate all a_{jk} and obtain the nonlinear system in five parameters: λ_1, λ_2, H and two arbitrary coefficients of the Laurent-series solutions.

5) Solve the obtained system.

This method has a few preferences. The first preference is that one does not need to transform system (8) to the single differential equation either in y or in x. Moreover at $C = -16/5$ not x, but x^2 may be an elliptic function. To construct the Laurent series for x^2 is easier than to find the fourth order equation in x^2. The main preference of this method is that the number of unknowns in the resulting algebraic system does not depend on number of coefficients of the first order equation. For example, eq. (10) with $m = 8$ includes 60 unknowns a_{jk}, and it is not possible to use the traditional way to find similar solutions. Using this method we obtain (independently of the value of m) a nonlinear algebraic system in five variables. It is important that all these calculations can be automatized due to computer algebra systems.

The first computer algebra realization has been written in AMP [17] by R. Conte. His algorithm bases on the α–method of the Painlevé test. Our Maple realization bases on transformations of the Laurent series [2].

The traditional way has one important preference. It allows to obtain solutions for an arbitrary C, because one has not to fix value of C to construct the Laurent series solutions.

The resulting nonlinear algebraic system can be solved using the standard Gröbner basis method. To obtain the explicit form of the elliptic function, which satisfies the known first order ODE, one can use the classical method due to Poincaré, which has been implemented in Maple [18] as the package "algcurves" [19].

5   CONCLUSION

The Painlevé test is a very useful tool to find single-valued solution in the analytic form. The procedure can be automatized. The corresponding computer algebra algorithm has been constructed in Maple [2]. Consideration of the motion equation on complex (time) plane and the use of the Painlevé test can assist to find a type of polynomial potential, which corresponds to solution with required type of singularities.

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