The Gauge Invariant ERG

O.J. Rosten[†] [*], T.R. Morris[‡] and S. Arnone[§]

April 26, 2005

[†] [‡] *School of Physics & Astronomy, University of Southampton, Highfield, Southampton S017 1BJ, U.K.*

[§] *Dipartimento di Fisica, Università di Roma "La Sapienza", P.le Aldo Moro, 2 - 00185 Roma - Italy*

Emails: [†]O.J.Rosten@soton.ac.uk, [‡]T.R.Morris@soton.ac.uk and [§]Stefano.Arnone@roma1.infn.it.

Abstract

We sketch the construction of a gauge invariant Exact Renormalization Group (ERG). Starting from Polchinski's equation, the emphasis is on how a series of ideas have combined to yield the gauge invariant formalism.

A novel symmetry of the ERG allows the flow equation to be modified, in such a way that it is suitable for the computation of the (universal) two-loop β-function. This computation has now been performed, within the framework of the ERG and, as such, in a *manifestly gauge invariant* way for the very first time.

1 Introduction

Two of the most powerful ideas in quantum field theory (QFT) are those of Wilson's renormalization group (RG) [2] and gauge symmetries. In this

[*]Based on a talk given by this author at Quarks-2004.

report, we describe a framework which incorporates both and, as such, may provide a powerful tool for the investigation of a host of problems within gauge theory. It is particularly noteworthy that, within this set-up, gauge invariance is manifest, allowing calculations to be done, without fixing the gauge [6].

We will concern ourselves with exact formulations of the renormalization group [2]–[4] which provide an equation describing how the effective action at some cutoff scale varies with this scale. To build toward a gauge invariant ERG, we begin by looking at scalar field theory, in section 2. Starting from Polchinski's equation [3], we generalise it to bring it to a form suitable for the construction of a gauge invariant ERG.

Applying the methods of the ERG to gauge theories was long beset by the difficulty of introducing a regulator which suppresses modes above Λ, whilst maintaining gauge invariance. In section 3, we begin by describing a regulator for Yang-Mills theory, suitable for use within the ERG [1]. Next we discuss a novel symmetry of the regularization scheme, which proves crucial in the generalisation of the ERG to a form suitable for computing the two-loop β-function.

The gauge invariant ERG of ref. [6] is introduced in section 4. After outlining its construction, we discuss the criterion for obtaining the universal value for β_1 and β_2. This leads us to a modified form of the flow equation. Finally, we outline how β-function coefficients can be extracted.

2 Generalising Polchinski's Equation

The key elements of the ERG are depicted in figure 1 [4]. Starting with the bare theory—for the scalar field φ—defined at the scale Λ_0, and parametrized by the bare action $S^{\mathrm{tot}}_{\Lambda_0}$, we integrate out degrees of freedom, to some intermediate scale, Λ. This scale separates the high energy modes from the low energy modes and can be viewed in one of two ways: for the high energy modes, it acts as an infra-red (IR) regulator whereas, for the low energy modes, it acts as an ultra-violet (UV) regulator. In recognition of this interpretation, we introduce two cutoff functions, $C_{\mathrm{IR}}(p^2/\Lambda^2)$ and $C_{\mathrm{UV}}(p^2/\Lambda^2)$, each of which cuts off modes in the region indicated by the subscript. We leave these cutoff functions general, demanding only that

$$C_{\mathrm{IR}}(p^2/\Lambda^2) + C_{\mathrm{UV}}(p^2/\Lambda^2) = 1$$

and that

$$\begin{aligned} C_{\rm UV}(0) &= 1; \\ \lim_{z\to\infty} C_{\rm UV}(z) &\to 0 \text{ (fast enough)}. \end{aligned}$$

In the high energy and low energy regions, we now modify the propagators by multiplying by the cutoff functions:

$$1/p^2 \to \begin{cases} \Delta_{UV} = \frac{C_{\rm UV}(p^2/\Lambda^2)}{p^2} & \text{low energy} \\ \Delta_{IR} = \frac{C_{\rm IR}(p^2/\Lambda^2)}{p^2} & \text{high energy.} \end{cases}$$

By performing the integral over high energy modes, we can obtain Polchinski's equation [3] for the flow of the interaction part of the Wilsonian effective action, $S^{\rm int}_\Lambda$:

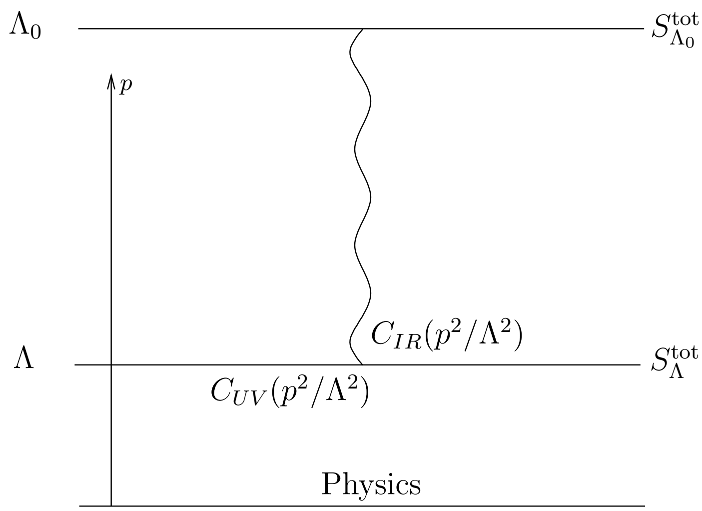

Figure 1: Integrating out modes.

$$\frac{\partial}{\partial\Lambda} S^{\rm int}_\Lambda[\varphi] = \frac{1}{2}\frac{\delta S^{\rm int}_\Lambda}{\delta\varphi}\cdot\frac{\partial\Delta_{UV}}{\partial\Lambda}\cdot\frac{\delta S^{\rm int}_\Lambda}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\frac{\partial\Delta_{UV}}{\partial\Lambda}\cdot\frac{\delta S^{\rm int}_\Lambda}{\delta\varphi}, \tag{1}$$

where, for the functions $f(x)$ and $g(y)$ and a momentum space kernel $W(p^2/\Lambda^2)$,

$$f\cdot W\cdot g = \iint d^Dx\, d^Dy\, f(x) W_{xy}\, g(y), \tag{2}$$

with

$$W_{xy} = \int d^D p \, W(p^2/\Lambda^2) e^{ip.(x-y)}. \tag{3}$$

There are three crucial properties of the Polchinski equation:

1. the flow in S corresponds to integrating out higher energy modes;
2. the partition function is left invariant under the flow;
3. it is defined non-perturbatively.

The point that will ultimately enable us to construct a gauge invariant flow equation is that other flow equations have these properties; indeed, from now on, we take these properties (in particular the first two) to define what is meant by a flow equation [6, 8].

The first step in the necessary generalisation of Polchinski's equation is trivial: we change variables in order to compute the flow of the total Wilsonian effective action, S. To this end, our regularised kinetic term is

$$\hat{S} = \frac{1}{2}\varphi \cdot \Delta_{UV}^{-1} \cdot \varphi,$$

and so $S = \hat{S} + S_\Lambda^{\mathrm{int}}$. We will call $\hat{S}$ the *seed action*. Introducing the new variable

$$\Sigma = S - 2\hat{S},$$

and defining $\dot{X} = -\Lambda\partial_\Lambda X$, we can rewrite Polchinski's equation, up to a discarded vacuum energy term, as

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta\varphi} \cdot \dot{\Delta} \cdot \frac{\delta\Sigma}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi} \cdot \dot{\Delta} \cdot \frac{\delta\Sigma}{\delta\varphi}.$$

Now, however, we perform a non-trivial step: we allow $\hat{S}$ to become very general [6] but keep S as the total Wilsonian effective action. In other words, the seed action is now defined independently of the total Wilsonian effective action. The question now is whether our flow equation is still valid. The partition function remains invariant under the flow:

$$\Lambda\frac{\partial}{\partial\Lambda}e^{-S} = -\frac{1}{2}\frac{\delta}{\delta\varphi} \cdot \dot{\Delta} \cdot \left(\frac{\delta\Sigma}{\delta\varphi}e^{-S}\right).$$

Moreover, the flow still corresponds to integrating out:

We can ensure that the flow equation is regularized, so all momentum integrals are bounded. Λ is the UV cutoff: momenta larger than some scale q must vanish in the limit $q/\Lambda \to \infty$. As $\Lambda \to 0$ all remaining contributions from any non-vanishing momentum scale disappear. *But* the physics is invariant under the flow. Contributions from a given momentum scale must be encoded in the effective action: we have integrated out!

As our flow equation satisfies the requisite requirements, it is perfectly valid. This generalisation is one of the necessary changes required for Yang-Mills theory [6]. Moreover, the great freedom in the seed action actually allows us to design it in order to simplify calculations!

3 Regularization for Yang-Mills Theory

3.1 The Construction

The final ingredient for our gauge invariant ERG is a regulator for Yang-Mills theory, based on a cutoff. To ensure compatibility with gauge invariance we cast equation 2 in gauge invariant form, replacing f and g with any two matrix representations, u and v:

$$\begin{aligned} u\{W\}v = \sum_{m,n=0}^{\infty} \int & d^D x\, d^D y\, d^D x_1 \cdots d^D x_n\, d^D y_1 \cdots d^D y_m \\ & W_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(x_1,\ldots,x_n;y_1,\ldots,y_m;x,y) \\ & \mathrm{tr}\left[u(x)A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)v(y)A_{\nu_1}(y_1)\cdots A_{\nu_m}(y_m)\right], \end{aligned} \tag{4}$$

where $\{W\}$, the 'wine', is the gauge covariantization of the kernel W_{xy} and equation 4 defines what we mean by the wine vertices. Note that the case $m = n = 0$ is just the original kernel W_{xy}, which we call a zero-point wine. We refer to the remaining higher-point wine vertices as 'decorations' of the zero-point wine.

We can now write the Yang-Mills kinetic term as

$$\frac{1}{2} F_{\mu\nu}\{c^{-1}\}F_{\mu\nu},$$

where c is a (UV) cutoff function and $\{c^{-1}\}$ the covariantization of its inverse.

However, this turns out to be insufficient to regularise the theory. The solution we choose is to embed our $SU(N)$ gauge theory in an $SU(N|N)$ gauge theory, which has sufficiently improved UV properties to ensure finiteness [1].

Taking the $SU(N|N)$ gauge field to be $\mathcal{A}_\mu$, we embed our physical $SU(N)$ field A^1_μ using the defining representation:

$$\mathcal{A}_\mu = \mathcal{A}^0_\mu 1 + \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix}. \tag{5}$$

The supermatrix possesses bosonic block diagonal fields, A^i, fermionic off-diagonal fields B, $\bar{B}$ and the central term $\mathcal{A}_0 1$. As required by $SU(N|N)$, the superfield is supertraceless: $\mathrm{str}\,\mathcal{A}_\mu = \mathrm{tr}\,A^1_\mu - \mathrm{tr}\,A^2_\mu = 0$. The covariant derivative is simply

$$\nabla_\mu = \partial_\mu - i\mathcal{A}_\mu,$$

where, in readiness for section 4, we have absorbed the coupling constant into the gauge field. The field strength is $\mathcal{F}_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]$ and the regularised kinetic term just

$$\frac{1}{2}\mathcal{F}_{\mu\nu}\{c^{-1}\}\mathcal{F}_{\mu\nu}.$$

Finally, we introduce the superscalar field, $\mathcal{C}$, to spontaneously break the $SU(N|N)$ symmetry in the fermionic directions. The fermionic fields acquire a mass, of the order the cutoff, and act as a set of Pauli-Villars (PV) fields. The problem of overlapping divergences, typical of PV regularisation, never arises as the covariant cutoff regularisation applies to all fields.

Equation 4 must now be modified to take account of the change from gauge invariance to supergauge invariance and the presence of $\mathcal{C}$s. For the former, we simply substitute $\mathcal{A}$s for As and replace the trace in equation 4 with a supertrace; for the latter, we now allow (up to some restrictions) decoration with both $\mathcal{A}$s and $\mathcal{C}$s [6].

Being supergauge invariant, the action has an expansion in terms of supertraces and products of supertraces. Suppressing position arguments and Lorentz indices we have:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{s_n} S^{X_1\cdots X_n} \mathrm{str}\,(X_1\cdots X_n) \\ &+ \frac{1}{2!}\sum_{m,n=1}^{\infty} \frac{1}{s_n s_m} S^{X_1\cdots X_n, Y_1\cdots Y_m} \mathrm{str}\,(X_1\cdots X_n)\,\mathrm{str}\,(Y_1\cdots Y_m) + \cdots \end{aligned}$$

where the X_i are $\mathcal{A}_\mu$ or $\mathcal{C}$. Only one cyclic ordering of each list appears in the sum; if a list is invariant under some nontrivial cyclic permutation, then s_n (s_m) is the order of the subgroup.

3.2 No-$\mathcal{A}^0$ Symmetry

The structure of $SU(N|N)$ leads to a symmetry concerning the field $\mathcal{A}^0$. The generator associated with $\mathcal{A}^0$ is 1 and, since this commutes with everything, $\mathcal{A}^0$ does not have a kinetic term. Were $\mathcal{A}^0$ to appear anywhere else in the action, it would act as a Lagrange multiplier, enforcing a non-linear constraint on the theory. We thus demand that the action is independent of $\mathcal{A}^0$. The theory is now invariant under a local 'no-$\mathcal{A}^0$' symmetry: $\delta\mathcal{A}^0_\mu(x) = \lambda_\mu(x)$. A full understanding of this symmetry has proven vital in performing calculations beyond one-loop [7].

4 A Gauge Invariant ERG

4.1 Construction

We start with an Abelian theory and make the simplest possible generalisation of the Polchinski equation:

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta A_\mu}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta A_\mu} - \frac{1}{2}\frac{\delta}{\delta A_\mu}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta A_\mu}.$$

This still has all the features of a flow equation that we want but we have the added benefit of not having fixed the gauge.

When we generalise to non-Abelian gauge theory, the exact preservation of the form of gauge transformations

$$\delta A_\mu = [D_\mu, \omega]$$

implies that $D_\mu = \partial_\mu - igA_\mu$ cannot renormalize [5]. We make use of this by changing variables: $A_\mu \to A_\mu/g$, $\hat{S} \to \hat{S}/g^2$. The first change means that $D_\mu = \partial_\mu - iA_\mu$ giving the nice result that the gauge field does not renormalize. Implementing both of our changes modifies the flow equation. One part takes the same form as before, but there is a second term which is not manifestly gauge invariant. Nonetheless, we can simply drop this term: the flow equation still leaves the partition function invariant and still corresponds to integrating out [6]. Our Abelian flow equation becomes:

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta A_\mu}\cdot\dot{\Delta}\cdot\frac{\delta \Sigma_g}{\delta A_\mu} - \frac{1}{2}\frac{\delta}{\delta A_\mu}\cdot\dot{\Delta}\cdot\frac{\delta \Sigma_g}{\delta A_\mu}$$

where $\Sigma_g = g^2 S - 2\hat{S}$.

It is now straightforward to generalise to the non-Abelian case: we must simply covariantize the cutoff and incorporate the $SU(N|N)$ regularization. The fully regularized non-Abelian flow equation is:

$$\dot{S} = a_0[S, \Sigma_g] - a_1[\Sigma_g] \tag{6}$$

where

$$a_0[S,\Sigma_g] = \frac{1}{2}\frac{\delta S}{\delta \mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} + \frac{1}{2}\frac{\delta S}{\delta \mathcal{C}}\{\dot{\Delta}^{\mathcal{C}\mathcal{C}}\}\frac{\delta \Sigma_g}{\delta \mathcal{C}}$$

and

$$a_1[\Sigma_g] = \frac{1}{2}\frac{\delta}{\delta \mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{A}\mathcal{A}}\}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} + \frac{1}{2}\frac{\delta}{\delta \mathcal{C}}\{\dot{\Delta}^{\mathcal{C}\mathcal{C}}\}\frac{\delta \Sigma_g}{\delta \mathcal{C}}.$$

The renormalized coupling, g, of the $SU(N)$ gauge field A^1_μ is defined through the renormalization condition:

$$S[\mathcal{A} = A^1, \mathcal{C} = \bar{\mathcal{C}}] = \frac{1}{2g^2}\mathrm{tr}\int d^D x\,(F^1_{\mu\nu})^2 + \cdots \tag{7}$$

where $\bar{\mathcal{C}}$ is the *vev* of the field $\mathcal{C}$ and the ellipsis denote higher dimension operators.

To complete the set up, we work in the broken phase and determine the zero-point wines, $\dot{\Delta}^{XX}$. The fields X can be any of the broken phase fields. The current flow equation cannot distinguish between A^1 and A^2 and so we treat them together, within the block-diagonal field A. The first step is to expand the flow equation for small coupling. We have the following expansions:

$$\begin{aligned} S &= 1/g^2 S_0 + S_1 + g^2 S_2 + \cdots \\ \Lambda\frac{\partial g}{\partial \Lambda} &= \beta_1 g^3 + \beta_2 g^5 + \cdots \end{aligned} \tag{8}$$

and, introducing $\Sigma_n = S_n - 2\hat{S}_n$, the weak coupling flow equations:

$$\begin{aligned} \dot{S}_0 &= a_0[S_0, \Sigma_0] \\ \dot{S}_1 &= -2\beta_1 S_0 + a_0[S_1,\Sigma_0] + a_0[S_0,\Sigma_1] - a_1[\Sigma_1] \\ &\vdots \end{aligned} \tag{9}$$

Now we determine the two-point tree level seed action vertices by using Lorentz invariance, dimensions and the renormalization condition. In this way, we determine that [6]

$$\hat{S}^{\ AA}_{0\mu\nu}(p) = 2\Box_{\mu\nu}(p)/c_p,$$

where the renormalization condition demands that $c(0) = 1$.

Using equation 9 to compute the flow of the two-point tree level Wilsonian effective action vertex $S^{\ AA}_{0\mu\nu}(p)$, we obtain an equation relating this to the corresponding seed action vertex and $\dot{\Delta}^{AA}$. By choosing to set $S^{\ AA}_{0\mu\nu}(p) = \hat{S}^{\ AA}_{0\mu\nu}(p)$, we find that $\dot{\Delta}^{AA} = c'_p/\Lambda^2$. In a similar fashion, we can compute the remaining zero-point wines.

This is the set-up used in ref. [6] to compute the one-loop β-function, without gauge fixing.

4.2 Universality of β_1 and β_2

Given the coupling g, within our renormalization scheme, we can perturbatively relate this to that of another renormalization scheme, $\tilde{g}$:

$$1/\tilde{g}^2 = 1/g^2 + \gamma + \mathcal{O}(g^2)$$

where γ is a matching coefficient. From equation 8 we have:

$$\tilde{\beta}_1 + \tilde{\beta}_2 g^2 = \beta_1 + \beta_2 g^2 - \dot{\gamma} + \mathcal{O}(g^4).$$

Agreement between the first two β-function coefficients is guaranteed, so long as γ does not run at either the tree or one-loop level.

Incorporation of PV fields into the ERG in fact generically introduces tree-level running. However, this can be removed by suitable choice of the seed action [6], guaranteeing universality at one-loop. To ensure universality at two-loops, there must be no hidden one-loop running couplings. Suitable tuning of the seed action is sufficient to remove all but one [7].

Referring back to equation 5, we see that, in addition to our physical gauge field A^1_μ, we also have the unphysical gauge field A^2_μ. This carries its own coupling g_2, which renormalizes in a different way from g. To obtain the universal value for β_2, we must isolate the effects of g_2, within the ERG. Then, at the end of a computation, we will tune g_2 to zero.

4.3 The New Flow Equation

To isolate the effects of g^2, we modify the flow equation to allow it to distinguish between A^1 and A^2. This must be done whilst, crucially, satisfying no-$\mathcal{A}^0$ symmetry. Introducing the new covariantized kernel $\{\dot{\Delta}^{\mathcal{AA}}_{\sigma}\}$, the flow equation receives the following modifications:

$$
\begin{aligned}
a_0[S,\Sigma_g] \quad \rightarrow \quad & a_0[S,\Sigma_g] \\
& + \frac{1}{16N}\frac{\delta S}{\delta \mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{AA}}_{\sigma}\} \times \\
& \qquad \times \left[\left\{\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}\right\} \mathrm{str}\,\mathcal{C} - 2\mathcal{C}\mathrm{str}\left(\mathcal{C}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}\right)\right] \\
& + \left[\left\{\mathcal{C}, \frac{\delta S}{\delta \mathcal{A}_\mu}\right\} \mathrm{str}\,\mathcal{C} - 2\mathcal{C}\mathrm{str}\left(\mathcal{C}\frac{\delta S}{\delta \mathcal{A}_\mu}\right)\right] \times \\
& \qquad \times \{\dot{\Delta}^{\mathcal{AA}}_{\sigma}\}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} \\
a_1[\Sigma_g] \quad \rightarrow \quad & a_1[\Sigma_g] \\
& + \frac{1}{16N}\frac{\delta}{\delta \mathcal{A}_\mu}\{\dot{\Delta}^{\mathcal{AA}}_{\sigma}\}\left\{\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}\mathrm{str}\,\mathcal{C} - 1\mathrm{str}\left(\mathcal{C}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}\right)\right\} \\
& + \frac{1}{16N}\left\{\mathcal{C}, \frac{\delta}{\delta \mathcal{A}_\mu}\mathrm{str}\,\mathcal{C} - 1\mathrm{str}\left(\mathcal{C}\frac{\delta}{\delta \mathcal{A}_\mu}\right)\{\dot{\Delta}^{\mathcal{AA}}_{\sigma}\}\right\}\frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}.
\end{aligned}
$$

Taking account of the running of g_2, the weak coupling flow equations are modified, and the two-point tree level seed action vertices change; hence we must recompute the zero-point wines [7].

4.4 Computing β-function Coefficients

With the formalism now in place, we can calculate β-function coefficients. By computing the flow of $S^{\,A^1 A^1}_{n\,\mu\;\nu}(p)$ and utilising the renormalization condition, we obtain an algebraic expression for β_n, whose value can be extracted via iterated use of the new flow equations.

Despite the changes to the flow equations, the diagrammatic methods of ref. [6] can still be employed. Indeed in ref. [7] they have been greatly enhanced, facilitating the computation of β_2. As anticipated, we find that, in the $g_2 \rightarrow 0$ limit, we regain the expected, universal coefficient demonstrating, beyond all reasonable doubt, the consistency of the approach.

Acknowledgements I would like to thank the organisers for a wonderful conference, and the Southampton University Development trust for financial support.

References

[1] S. Arnone, Y. A. Kubyshin and T. R. Morris, J. F. Tighe, Int. J. Mod. Phys. A17 (2002) 2883.

[2] K. Wilson and J. Kogut, Phys. Rep. 12C (1974) 75.

[3] J. Polchinski, Nucl. Phys. B231 (1984) 269.

[4] T. R. Morris, Prog. Theor. Phys. Suppl. 131 (1998) 395, in *New Developments in Quantum Field Theory*, NATO ASI series 366, (Plenum Press, 1998), Int. J. Mod. Phys. B12 (1998) 1343.

[5] T. R. Morris, Nucl. Phys. B573 (2000) 97.

[6] S. Arnone, A. Gatti and T. R. Morris, Phys. Rev. D67:085003,2003.

[7] O.J. Rosten, T. R. Morris and S. Arnone, in preparation.

[8] J.I. Latorre and T. R. Morris, Int.J.Mod.Phys.A16:2071-2074,2001