Geometry of fast moving strings.

A. Mikhailov

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1 Introduction.

In this talk I will discuss some recent progress in AdS/CFT correspondence. The AdS/CFT correspondence is a strong-weak coupling duality. Four-dimensional $N = 4$ supersymmetric Yang-Mills theory with the coupling constant g_{YM} and 'tHooft coupling $\lambda = g_{YM}^2 N$ is conjectured to be equivalent to the Type IIB string theory on $AdS_5 \times S^5$ with the radius $R \sim \lambda^{1/4}$ and string coupling constant $g_{str} = g_{YM}^2$. This means that weakly coupled Yang-Mills (small λ) is mapped to the string theory on highly curved AdS space. When AdS space is highly curved, the string worldsheet theory becomes strongly coupled.

Therefore, the weakly coupled Yang-Mills maps to the strongly coupled string worldsheet theory, and vice versa. Nevertheless, some elements of the YM perturbation theory were recently reproduced from the string theory side.

The most recent example are operators with the large R-charge corresponding to the spinning strings in $AdS_5 \times S^5$ [1, 2, 4, 5]. Consider the operators of the form

$$\mathrm{tr}\ \underbrace{\phi\phi\ldots\partial\ldots\partial\phi\ldots}_{L} \tag{1}$$

It turns out that in some perturbative calculations the small parameter is $\frac{\lambda}{L^2}$ rather than λ. This allows to use the Yang-Mills perturbation theory in the regime where λ is large, where we can also compute on the string theory side.

2 Coherent states.

It is convenient to consider instead of the single-trace operators tr $\phi \cdots \phi$ the corresponding state in the theory on $\mathbf{R} \times S^3$. This is a chain of one-particle states ("partons"). We will be interested mainly in the one-loop computations, so we will consider partons as 1-particle states in the free theory. These 1-particle states form a representation of the conformal group $SO(2,4) \simeq SU(2,2)$ known as the "singleton representation". The conformal group $SU(2,2)$ together with the R-symmetry group $SU(4)$ form a bosonic part of the super-group of super-conformal transformations, called $PSU(2,2|4)$. The 1-particle states are in the "super-singleton" representation of this group.

2.1 Definition of coherent states.

We will now define a a special class of 1-particle states known as "coherent states". We start with the "basic"state:

$$\psi_{\mathbf{1}} = \int_{S^3} d^3\vec{n}(\Phi_1(\vec{n}) - i\Phi_2(\vec{n}))|0> \tag{2}$$

Here Φ_1 and Φ_2 are two of the six scalar fields of the $N = 4$ super-Yang-Mills theory and $|0>$ is the conformally invariant vacuum of this theory. Our state $\psi_{\mathbf{1}}$ can be described as the creation operator of the zero harmonic of the field $\Phi_1 - i\Phi_2$ on S^3, acting on the vacuum.

Let us act on this state $\psi_{\mathbf{1}}$ by the superconformal group $PSU(2,2|4)$. Define ψ_g:

$$\psi_g = g.\psi_{\mathbf{1}} \tag{3}$$

The stabilizer of $\psi_{\mathbf{1}}$ is $PSU(2|2) \times PSU(2|2) \times U(1)^2$, so the coherent states are parametrized by the coset space

$$\frac{PSU(2,2|4)}{PSU(2|2) \times PSU(2|2) \times U(1)^2} \tag{4}$$

This definition of the coherent states is the generalization of the coherent states in the $SU(2)$ sector introduced in [6].

2.2 Explicit formulas.

Let us start with acting on $\psi_{\mathbf{1}}$ by the subgroup $SO(6) \subset PSU(2,2|4)$. We get:

$$\int_{S^3} d^3\vec{n}(Z_1\Phi_1 + \ldots + Z_6\Phi_6)|0> \tag{5}$$

where $Z_i \in \mathbf{C}$, $Z_1^2 + \ldots Z_6^2 = 0$. These states are zero harmonics on S^3 of the creation operator of the complex scalar $Z_1\Phi_1 + \ldots + Z_6\Phi_6$, acting on the vacuum. Now let us characterize the orbit of $\psi_{\mathbf{1}}$ under $SU(2,2) \times SU(4) \subset PSU(2,2|4)$. Take the field $Z_1\Phi_1 + \ldots + Z_6\Phi_6$ but instead of zero harmonic on S^3, create the following positive frequency solution:

$$\frac{1}{Y_{-1}\cos\tau + Y_0\sin\tau - (\vec{Y}\cdot\vec{n})} \tag{6}$$

where again Y_i are the complex numbers, with the constraint $Y_{-1}^2 + Y_0^2 - Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 = 0$ and $\tau, \vec{n}$ are the coordinates on $\mathbf{R} \times S^3$. The manifold of the complex lightlike vectors Y in $\mathbf{C}^{2+4}$ consists of two connected components. Those Y which can be rotated by $SO(2,4)$ to $(1,-i,0,0,0,0)$ belong to the first component, and those which can be rotated to $(1,i,0,0,0,0)$ belong to the second component. We need only the first component; the second component will give negative-frequency solutions. Such states form an orbit of $\psi_{\mathbf{1}}$ under $SU(2,2) \times SU(4) \subset PSU(2,2|4)$.

The states ψ_g generate the whole super-singleton representation (although there are some linear relations among them). Consider chains of coherent states of partons:

$$\mathrm{tr}\ \psi_{g_1} \otimes \psi_{g_2} \otimes \cdots \otimes \psi_{g_L} = \Psi_{[g(n)]} \tag{7}$$

Such states generate the space of L-particle states.

2.3 Continuous limit of the spin chain.

Conjecture. When $L \to \infty$ and $\frac{\lambda}{L^2}$ is small, the dynamics (renormgroup evolution) of the operator is described by a classical system with the phase space — the space of contours $g(\sigma)$ where $\sigma = n/L$. The symplectic form is

$$\Omega = d\alpha, \quad \alpha = (\overline{\Psi}_{[g(\sigma)]}, d\Psi_{[g(\sigma)]}) \tag{8}$$

The Hamiltonian is

$$H = (\overline{\Psi}_{[g(\sigma)]}, H_{int}^{YM}\Psi_{[g(\sigma)]}) \tag{9}$$

To motivate this proposal, notice that if $g(\sigma) = g = \text{const}$ then $\psi_g \otimes \cdots \otimes \psi_g$ is a ferromagnetic vacuum. The operators $\Psi_{g(\sigma)}$ with continuous $g(\sigma)$ correspond to the classical long wavelength excitations about this ferromagnetic vacuum.

A calculation [7] using the one-loop dilatation operator computed in [8, 9] gives

$$H_{cl}^{1-loop} = \int d\sigma ||\partial_\sigma g(\sigma)||^2 \tag{10}$$

where $||dg||^2$ is the invariant metric on $\frac{PSU(2,2|4)}{PSU(2|2)\times PSU(2|2)\times U(1)^2}$.

3 View from the string theory side.

3.1 Equivalence of two dynamical systems.

The dimension of the coset $\frac{PSU(2,2|4)}{PSU(2|2)\times PSU(2|2)\times U(1)^2}$ is 16|16 (sixteen even and sixteen odd coordinates). Therefore the contour $g(\sigma)$ is specified by 16 even and 16 odd functions of one real variable.

But the classical string solutions in $AdS_5 \times S^5$ are also specified by 16 even and 16 odd functions of one real variable. Indeed, fix the global time $T = 0$; the classical string is specified by its position (shape) at $T = 0$ and its velocity. Eight functions are needed to determine the position of the string at $T = 0$, and another eight functions to determine the velocity. Therefore, the phase space is parametrized by 16 functions. In addition, we need to specify 32 fermions, half of which can be gauged away by the kappa-symmetry; therefore, we need also 16 odd functions.

This rough counting shows that the continuous spin chain and the string worldsheet sigma-model have the same number of degrees of freedom. This allows us to conjecture that *the classical worldsheet sigma-model for the Type IIB superstring on $AdS_5 \times S^5$ is equivalent as a Hamiltonian system to the classical parton chain.* (More precisely, we need the union of the spin chains for all the possible values of the length.)

For the classical worldsheet theory to be valid, we need at least $\lambda >> 1$. For the YM perturbation theory to work, we need $\lambda/L^2 << 1$ (this is a conjecture). Therefore, we need very large L.

Large number of partons means that the state has large R-charge, or from the point of view of the string theory the large momentum in S^5. Therefore *large L corresponds to the fast moving strings.*

In the limit $L = \infty$ (for fixed large λ) the string worldsheet becomes degenerate. In other words, every point on the string moves with the speed of light.

3.2 Degenerate surfaces and null surfaces.

The surface is called degenerate if the induced metric is degenerate. When the string moves very fast, the worldsheet becomes a degenerate surface. The inverse is not quite true; not every degenerate surface can be obtained as a limit of the string worldsheet, when the velocity of the string approaches the speed of light. In fact, only a special class of the degenerate surfaces, called null-surfaces, can be obtained in this way (see for example [10]).

Definition. A null-surface is a degenerate surface ruled by the light rays. Light rays are also called null-geodesics.

There are two types of light rays in $AdS_5 \times S^5$, therefore the null-surfaces in $AdS_5 \times S^5$ can be of two types. The first type are null-surfaces ruled by the light rays totally inside AdS_5. Such null-surfaces extend to the boundary of AdS_5. They describe the propagation on the string worldsheet of the shock wave, originating from the cusp on the trajectory of the heavy spectator quark on the boundary. We will not discuss this type of the null-surfaces here.

What we need now is the second type of the null-surfaces, those which are generated by the light rays which are obtained as a diagonal in the product of a timelike geodesic in AdS_5 and an equator in S^5, see Fig. 1. Notice that the time-like geodesics on AdS_5 are intersections of planes $\mathbf{R}^{2+0} \subset \mathbf{R}^{2+4}$ with the hyperboloid representing AdS_5. Therefore the time-like geodesics in AdS_5 are parametrized by the coset space

$$\frac{SO(2,4)}{SO(2) \times SO(4)} \tag{11}$$

And equators of S^5 are parametrized by

$$\frac{SO(6)}{SO(2) \times SO(4)} \tag{12}$$

Therefore the null-surfaces in $AdS_5 \times S^5$ are specified by the contours in:

$$\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \tag{13}$$

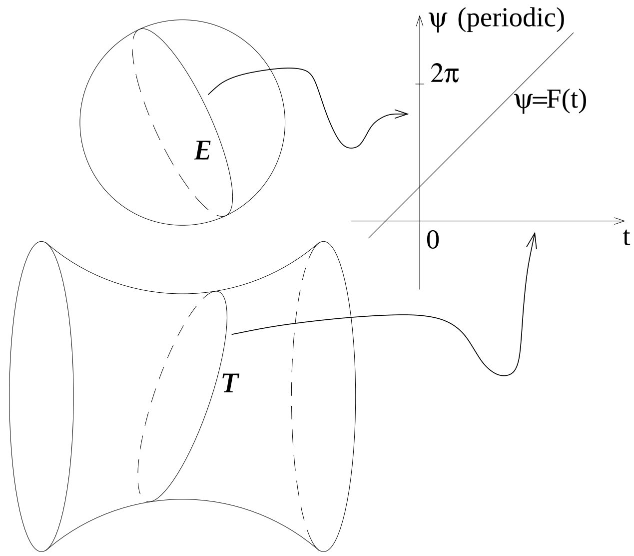

Figure 1: *A null-geodesic in* $AdS_5 \times S^5$ *is specified by the choice of an equator* $\mathbf{E}$ *in* S^5, *a time-like geodesic* $\mathbf{T}$ *in* AdS_5 *and a map* $F : \mathbf{T} \to \mathbf{E}$ *which maps the angular parameter* ψ *on the equator to the time* t *on the geodesic, up to a constant.*

This coset space is equivalent to:

$$\frac{SU(2,2)}{S(U(2)\times U(2))} \times \frac{SU(4)}{S(U(2)\times U(2))} \tag{14}$$

This is the bosonic part of the supercoset

$$\frac{PSU(2,2|4)}{PSU(2|2)\times PSU(2|2)\times U(1)^2} \tag{15}$$

The fermionic degrees of freedom on the worldsheet in the null-surface limit promote the bosonic coset (14) to the super-coset (15) [7]. The super-coset (15) can be thought of as a super-Grassmanian parametrizing the embeddings of the (2|2)-dimensional complex superspaces into a (4|4)-dimensional complex superspace. It is the known as the "(4,2,2) analytic superspace" in the supersymmetry literature. We can think of it as the super-symmetrization of the future tube of the Minkowski space [11].

We see that *the phase space of the classical parton chain is equivalent to the moduli space of supersymmetric null-surfaces as a manifold with the* *action of $PSU(2,2|4)$*. One can verify that the symplectic structures also agree.

What about the Hamiltonian?

It turns out that on the moduli space of null-surfaces, there is a natural Hamiltonian.

To define this Hamiltonian we have to study the behavior of the extremal surfaces approximating the null-surface.

Let us denote $\epsilon^2 = \lambda/L^2$. Consider the string worldsheet at finite very small ϵ^2 — denote it Σ. This is a nearly-degenerate extremal surface. This extremal surface is *locally* close to a null-surface, deviation $\sim \epsilon^2$. *Locally* means: Pick a point $x_0 \in \Sigma$; there exists a null-surface Σ_0 such that in the neighborhood of x_0 of the radius of the order R_{AdS} the coordinate distance between Σ and null Σ_0 is of the order ϵ^2. But Σ is close to Σ_0 only locally! If we follow the extremal surface Σ later in time, the deviation between Σ and Σ_0 will accumulate. After the time $\Delta T \sim \frac{1}{\epsilon^2}$ the difference between Σ and Σ_0 will be of the order 1. Then, Σ will approximate a different null-surface, which we call $\Sigma_0^{\Delta T}$.

Therefore we have a one-parameter family $\Sigma_0^{(\Delta T)}$ of null-surfaces. This is the slow evolution of the null-surface Σ_0.

This slow evolution has two important properties:

1. it does not actually depend on the choice of approximating extremal surface Σ, which we used for its definition[1];

2. it is a Hamiltonian flow on the moduli space of the null-surfaces.

Outline of the proof. Let Σ be an approximating extremal surface. We can choose on Σ a special set of coordinates (σ, τ) so that:

$$\begin{cases} (\partial_\tau x)^2 + \epsilon^2 (\partial_\sigma x)^2 = 0 \\ (\partial_\tau x, \partial_\sigma x) = 0 \end{cases} \tag{16}$$

These coordinates have a good property: the embedding functions $x(\sigma, \tau)$ have a nice limit when $\epsilon^2 \to 0$, describing the embedding of the null-surface Σ_0. In these coordinates the worldsheet action is:

$$S = \int d\tau d\sigma \left[(\partial_\tau x)^2 - \epsilon^2 (\partial_\sigma x)^2 \right] \tag{17}$$

[1]There is a subtlety here. In fact we should consider parametrized null-surfaces, and the approximating extremal surface Σ should be such that the spatial coordinate on Σ in the conformal gauge agrees with the parametrization of Σ_0. See [12] for details.

We can think of the term proportional to ϵ^2 as a small perturbation; the unperturbed action is:

$$S_0 = \int (\partial_\tau x)^2 d\tau d\sigma \tag{18}$$

It describes a continuous collection of the free massless particles ("string bits"). The unperturbed system has a remarkable property: *all its trajectories are periodic*. Null-surfaces in $AdS_5 \times S^5$ are periodic, just like solutions of the free massless equations.

We are in the following situation.

- Null surfaces are described by an integrable system H_0 which is highly resonant: all the trajectories are periodic.
- Passing from the null-surface to the extremal surface corresponds to the small perturbation $\epsilon^2 \Delta H$.

Because the original system was resonant, the small perturbation qualitatively changes the picture of the dynamics. The trajectories of the perturbed system are not periodic anymore, and the approximating periodic trajectory (the null-surface) slowly evolves ("secular drift").

The space of periodic orbits is a symplectic manifold. The long term evolution is a Hamiltonian flow on this manifold. The Hamiltonian of the slow evolution is given by averaging $\langle \Delta H \rangle$ of the perturbation over the periodic trajectory. In our case:

$$\Delta H = \int d\sigma (\partial_\sigma x)^2 \tag{19}$$

Therefore

$$\langle \Delta H \rangle = \int_0^{2\pi} d\tau \int d\sigma (\partial_\sigma x_0)^2 \tag{20}$$

where $x_0(\sigma, \tau)$ is the embedding of the null-surface. This formula depends only on $x_0(\sigma, \tau)$ — the null-surface (the choice of the approximating extremal surface does not enter). It is clear from this formula that the Hamiltonian of the slow evolution is a local functional of the null-surface.

We have seen that the null-surface is parametrized by a function $g(\sigma)$ with values in

$$\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \tag{21}$$

One can verify that

$$\langle \Delta H \rangle = \int d\sigma ||\partial_\sigma g(\sigma)||^2 \tag{22}$$

in agreement with the one-loop result (10) on the field theory side.

4 Conclusion.

YM operators with the large R-charge should correspond to the long classical strings on $AdS_5 \times S^5$. If we knew which operator corresponds to which classical string worldsheet, it would immediately give us $E(\lambda)$ exactly in λ. (As the energy of the string.) Unfortunately, we do not know a priori which operator corresponds to which worldsheet.

But still, we can say that two Hamiltonian systems (parton chain and classical string) are equivalent, as Hamiltonian systems. This is a very non-trivial statement!

Even though locally two Hamiltonian systems with the same dimension of the phase space are always equivalent, globally there are many obstructions.

We are comparing systems which are defined as perturbation series in some small parameter. In the order ϵ^2 the slow evolution is the invariant. We have shown that it matches the YM renormgroup evolution.

In the next order in ϵ^4, the monodromy over the period will give us H_2 mod $H_2 \sim H_2 + \{F, H_2\}$. Roughly speaking, the invariant will be the average of H_2 over the invariant tori of H_1. This already gives many invariants. But there are resonant tori therefore there should be more invariants.

Generally speaking we should try to match the integrable structure of YM to integrable structure of the Type IIB worldsheet. Important steps in this direction were made in [13, 14, 15].

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