On the numerical method of Casimir energy renormalization in the presence of logarithmical divergencies

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Abstract

A non-subtractive recipe of Casimir energy renormalization efficient in the presence of logarithmically divergent terms is proposed. It is demonstrated that it can be applied even then, when energy levels can be obtained only numerically and neither their asymptotical behavior, nor the analytic form of spectral equation is known. The results of calculations performed with this method are compared to those obtained by means of explicit subtraction of divergent terms from energy.

1 Introduction

Ever since Casimir [1] has obtained corrections to the energy of a macroscopic system due to vacuum fluctuations of quantized electromagnetic field in 1948 this effect has been intensively studied both from theoretical and experimental points of view. Nevertheless the calculation of Casimir energy except for the most simple problems involving free fields inside cavities with

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flat boundaries is quite non-trivial yet. To realize this, recall a great number of papers devoted to a free field in the interior of a sphere [2-11]. Despite the fact that in this case one can explicitly write out spectral equations, the first analytical results have been obtained only in [12] for massive scalar field and in [13] for fermions.

It should be stressed that the knowledge of analytical form of spectral equation has been crucial in [12] since it makes possible the transition from the sums containing the unknown energy levels to the integrals with the explicit integrands [14]. The goal of this paper is to demonstrate a method which can be applied to numerical calculation of Casimir energy in cases when these requirements are not met. Moreover, we are not going to use a rather standard trick [7], [15], which lets one overcome problems arising due to the presence of logarithmic divergency in Casimir energy of free massless fields inside spherical shells. Note that logarithmic divergency appears as a consequence of a curved surface bounding the shell and makes the energy renormalization ambiguous. The main idea of the trick is to consider the "inner" and "extra" problems together since their logarithmic divergencies cancel each other.

The ambiguity of Casimir energy renormalization in the presence of logarithmic divergency is quite obvious. Indeed, in case of massless fields the energy of the system can be characterized by a single dimensional parameter L which is the linear size of the system. The regularization parameter α can be also chosen to have a dimension of length. In the absence of logarithmic divergency the "minimal subtraction" of singular terms is not only natural but also well grounded. Indeed, any term proportional to α^{-s} $(s > 0)$ is obviously proportional to L^{s-1}, i.e. to the non-negative power of L. This makes it possible to normalize the final result at $L = \infty$, where Casimir energy should become zero, and subtract all singular terms at the same time. After such subtraction the only remaining term in the limit $\alpha \to 0$, which reads $c\alpha^0/L$, provides the final result.

In the presence of logarithmic divergency the subtraction becomes ambiguous, since in order to renormalize the term $c\alpha^0 \log(\alpha/L)/L$, one should subtract $c\alpha^0 \log(d\alpha/L)/L$, where d is an arbitrary constant, which cannot be determined from the normalizing condition at $L \to \infty$.

2 Massive scalar field in 1D

To illustrate the main idea of the proposed recipe let's consider Casimir energy with the logarithmic divergency in the most simple case, i.e. Casimir energy of massive scalar field on interval of length L with Dirichlet boudary conditions at the ends of the interval. It reads:

$$\mathcal{E}_{cas} = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n = \frac{1}{2}\sum_{n=1}^{\infty}\sqrt{(\pi n/L)^2 + m^2} \tag{1}$$

Note that we aren't going to subtract the Minkowski vacuum contribution from (1) as proposed in "standard" approaches. Instead we will directly pass to renormalizaton.

The regularization of (1) requires the introduction of the parameter α which has a dimension of length and stands in the argument of the cut-off function $F(\alpha\omega_n)$:

$$\mathcal{E}^{(r)}_{cas} = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n F(\alpha\omega_n) \tag{2}$$

Trivial considerations based on dimensional analysis lead to the following expression for the regularized Casimir energy

$$\mathcal{E}^{(r)}_{cas} \simeq c_{-2}\frac{L}{\alpha^2} + c_{-1}\frac{L^0}{\alpha^1} + c_0\frac{1}{L} + c_\lambda m^2 L\log(\alpha/L) + \cdots \tag{3}$$

It can be easily verified that for various cut-off functions such as $F(x) = \exp(-x)$, $F(x) = \exp(-x^2)$, $F(x) = \exp(-x^3)$, ..., $F(x) = \exp(-x^6)$, $F(x) = \exp(-2\cosh(x) + 2)$, ..., identical c_λ are obtained, while c_0 are different. The identity of c_λ for different $F(x)$ can be demonstrated with the following estimation for the sum giving rise to the logarithmic divergency:

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2}\frac{m^2}{\omega_n}F(\alpha\omega_n) \sim \frac{m^2 L}{4\pi}\log N \tag{4}$$

$$\sim \frac{m^2 L}{4\pi}\log(\Delta x L/\alpha) = \frac{m^2 L}{4\pi}\log(L/\alpha) + \frac{m^2 L}{4\pi}\log(\Delta x)\ ,$$

where $N \sim \Delta x L/\alpha$ and Δx is a cut-off interval of $F(x)$.

Since any subtraction in the presence of logarithmic divergency is ambiguous this procedure should be excluded from consideration along with the logarithmic divergency itself. To realize that let's calculate $\partial_L^2\mathcal{E}^{(r)}_{cas}$:

$$\partial_L^2 \mathcal{E}_{cas}^{(r)} \simeq c_0 \frac{2}{L^3} + c_\lambda \frac{m^2}{L} + \cdots \tag{5}$$

The obtained expression is regular in the limit $\alpha \to 0$, so no subtraction is required. The knowledge of the function $\partial_L^2 \mathcal{E}_{cas}^{(r)}$, lets one reconstruct the required $\mathcal{E}_{cas}^{(r)}$ unambiguously, since the initial conditions at $L \to \infty$ are well-known: both $\mathcal{E}_{cas}^{(r)}$ and $\partial_L \mathcal{E}_{cas}^{(r)}$ should become zero. Note that while (5) doesn't describe the asymptotical behavior of $\partial_L^2 \mathcal{E}_{cas}^{(r)}$ at $L \to \infty$, it demonstrates the disappearence of all singular terms in it. Moreover the following integral approximation shows that $\partial_L^2 \mathcal{E}_{cas}^{(r)}$ vanishes for $L \to \infty$:

$$\mathcal{E}_{cas}^{(r)} = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n F(\alpha \omega_n) = \tag{6}$$

$$= \frac{1}{4} \sum_{n=-\infty}^{\infty} \omega_n F(\alpha \omega_n) - \frac{m}{4} F(\alpha m) \approx$$

$$\approx \frac{1}{2} \int_{-\infty}^{\infty} dx (L/\pi) \sqrt{x^2 + m^2} F(\alpha \sqrt{x^2 + m^2}) - \frac{m}{4} F(\alpha m)$$

3 Method of calculation in general case

Let's generalize the proposed method in such a way that it doesn't require the analytical expression for energy levels. Suppose one has a set of energy levels of some spectrum ω_n and the corresponding Casimir energy contains the logarithmic divergency. First of all it turns out to be possible to modify the initial expression for the Casimir energy by introduction of some parameter μ in such a way that

$$\frac{1}{2} \sum_{n=1}^{\infty} \sqrt{\omega_n^2 - \mu^2} F(\alpha \sqrt{\omega_n^2 - \mu^2}) \tag{7}$$

doesn't contain the logarithmic divergency. For the massive scalar field on an interval μ is obviously equal to the mass of the field. In less trivial three-dimensional cases with spherical symmetry μ is some parameter having a dimension of mass which characterizes the total coefficient by the logarithmic divergency with all values of angular momentum taken into account.

The next step is to introduce an "additional mass" of the field $\mathcal{M}$ and study the Casimir as a function of it in the range from $\mathcal{M} = 0$ to $\mathcal{M} = \infty$.

In fact it's helpful to introduce another parameter M: $M^2 \equiv \mathcal{M}^2 + \mu^2$ and study the modified Casimir energy

$$\mathcal{E}_{cas}(M) = \frac{1}{2}\sum_{n=1}^{\infty}\sqrt{\omega_n^2 + M^2 - \mu^2} \tag{8}$$

as a function of M in the range from $M = \infty$ to $M = \mu$. To realize this one should calculate numerically the following quantity in the specified range of M:

$$\partial_M^2 \left(\mathcal{E}_{cas}^{(r)}(M)/M\right) = \tag{9}$$

$$= \partial_M^2 \left[\frac{1}{2}\sum_{n=1}^{\infty}\sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1}\;\; F\left(\alpha\sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1}\right)\right]$$

Note that in contrast to our previous considerations we have substituted dimensionless quantity

$$\frac{1}{M}\sqrt{\omega_n^2 + M^2 - \mu^2} = \sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1}$$

to the argument of $F(x)$, so that α should be also taken dimensionless.

An alternative interpretation of (9) follows from the observation that M acts as an effective length L in the expression for Casimir energy. Indeed, (9) can be obtained from the initial expression as a result of the folowing transformation of the spectrum. At the first step ω_n is transformed to $\omega_n' = \sqrt{\omega_n^2 - \mu^2}/\mu$, which is dimensionless spectrum with the subtracted effective mass. After that the scale transformation of the system $x \to x(M/\mu)$ and $\omega_n' \to \omega_n'' = \omega_n'/(M/\mu)$ is performed. In the end the unit mass is "added" to the obtained spectrum:

$$\omega_n'' \to \sqrt{(\omega_n')^2 + 1} = \sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1} \tag{10}$$

The limit $M \to \infty$ obviously corresponds to the infinite size of the system, while for $M = \mu$ one obtains the initial spectrum divided by μ.

It's easy to see that all divergent terms in (9) vanish. In the limit $n \to \infty$ one can make use of the following expansion

$$\sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1} \approx \frac{\sqrt{\omega_n^2 - \mu^2}}{M} + \frac{M}{2\sqrt{\omega_n^2 - \mu^2}} + \cdots \tag{11}$$

The first term in this expansion gives rise to the sum which is free of logarithmic divergency due to the definition of μ. Other divergencies are proportional to $(M/\alpha)^2/M$ and $(M/\alpha)^1/M$ and vanish when the second-order derivative is calculated. The second term leads to the logarithmic divergency with the coefficient proportional to M^1 by it, which also vanishes.

As a result one has (9) regular for $\alpha \to 0$ and the natural normalizing condition $\mathcal{E}_{cas}(M \to \infty) = 0$. The latter can be understood from two different points of view. On one hand the quantized field with the infinitely large mass should have zero Casimir energy. On the other hand the Casimir energy in the limit of the infinite size of the system should become zero. Whichever interpretation is chosen, the obtained results let one reconstruct the required $\mathcal{E}_{cas}(M = \mu)$ which corresponds to the initial spectrum.

Note, that principally one could consider $\mathcal{E}^{(r)}_{cas}(M)$ instead of $\mathcal{E}^{(r)}_{cas}(M)/M$. However that would increase the order of derivative required to exclude all divergent terms by one what is undesirable from the practical point of view.

The proposed method turns out to be efficient not only in the most trivial one-dimensional cases but also in more realistic three-dimensional ones. However to employ it in three-dimensional case one should inevitably calculate the fourth-order derivative of the Casimir energy (9) since the main singular term, which is proportional to the volume of the system, reads $c_{-4}L^3/\alpha^4$. It should be also noted that in this case the calculations of the sums become more sophisticated since the final value of a typical sum is about 40 orders lower than intermediate values obtained during its calculation and extra floating-point precision is required.

4 Numerical results

For scalar field on an interval $[0; L]$ with $L = 1$ spectrum reads

$$\omega_n = \sqrt{\frac{(\pi n)^2}{L^2} + m^2} \tag{12}$$

The result of the straightforward application of our method with various cut-off functions such as $F(x) = \exp(-x)$, $F(x) = \exp(-x^2)$, $F(x) = \exp(-x^3)$, $\ldots$, $F(x) = \exp(-x^6)$, $F(x) = \exp(-2\cosh(x) + 2)$ is presented on Fig.1. It has been shown that for each of these functions the same result is obtained and what's more the precision of coincidence depends only on the number of energy levels taken into account and the number of right digits used in the realization of floating-point arithmetics as well.

As to dependence of the Casimir energy on the mass of the field some important aspects should be stressed. First of all in the limit $m \to 0$ a well-known result for the massless scalar field is obtained. In the range of large values of m Casimir energy decreases exponentially as e^{-2mL} what could be expected from qualitative considerations. The results obtained with our method in this trivial case are in agreement with those obtained using the traditional subtractive technique.

To demonstrate how the method can be employed in less trivial cases we consider the massless scalar field inside of a spherical shell of radius $R = 1$. In this case the same set of cut-off functions has been used. The values of an effective mass $\mu = 0.1377$ obtained with each of these functions coincide up to the first four digits. Consequently the precision of the obtained $E_{cas} = 3.790 \cdot 10^{-3}$ has the same order, what corresponds to about 200 s-levels taken into account during the calculations. The number of energy levels taken into account is directly affected by the range which the regularization parameter α used in the calculations belongs to. Therefore one can control precision of the final result simply changing the range of employed values of the regularization parameter.

Note that in the framework of this approach we have obtained not only Casimir energy of the massless scalar field (corresponding to $\mathcal{M} = 0$) but also Casimir energy for all possible values of mass in the range from zero to the "effective" infinity. The dependence of the Casimir energy of the scalar field inside the sphere on the mass of the field is presented on Fig.2.

It should be stressed that on the contrary to the results obtained with the traditional subtractive technique in [12] our result doesn't contain logarithmical singularity at $\mathcal{M} = 0$ what seems more reasonable from physical point of view. The most likely explanation of this difference is that the subtractive procedure contains some arbitrariness. As a result some function which has regular behavior at $\mathcal{M} \to \infty$ but is singular at $\mathcal{M} \to 0$ could be subtracted from the final result.

As has been pointed out in [12,15] there is no argument at present which can remove this arbitrariness in case of a massless scalar field inside of a sphere. Therefore Casimir effect in the whole space with Dirichlet boundary conditions on the sphere is usually considered. It seems reasonable to calculate the Casimir energy in the last case employing our method.

In fact there are two ways to proceed to take exterior into account. The first one deals with the continuous spectrum and requires that all the regularized sums be replaced with the appropriate integrals containing the energy levels density in the integrand. The second way lets one work with discrete spectrum all the time. To realize that one should place the initial spherical shell into another sphere with the radius $R_{out} = kR_{in}$ where $k \geq 1$ and calculate the Casimir energy for the system bounded by the outer sphere taking into account boundary conditions on both of the spheres. For each finite k the spectrum is discrete and the developed technique can be applied without modification. The required result can be achieved in the limit $k \to \infty$. In practice it turns out that for $k \geq k_0$, where k_0 is finite and depends on the required precision, the result doesn't depend on k. It turned out that in the considered case in order to get 4 right digits in the final result $k_0 \simeq 10$ is quite enough.

The final result of the calculations is presented on Fig.3. Note that while the qualitative behavior of Casimir energy is the same as that obtained with methods employing explicit subtraction [15], there is no absolute coincidence. For example, for the massless field the result obtained with our recipe is $\mathcal{E}_{cas}(M = 0) = 0.0039$ while direct subtraction leads to $\mathcal{E}_{cas}(M = 0) = 0.0028$.

5 Conclusion

To summarize, an efficient technique for numerical calculation of Casimir energy in the presence of logarithmical divergencies has been developed. The advantages of the proposed method are its ideological triviality and universality which let one apply it to a wide range of problems in which numerical values for energy levels can be obtained. The results of its application to a number of problems appear to be reasonable, especially in case of a curved boundary. As to disadvantages they are purely technical: one should employ floating-point arithmetics with extra precision to carry out calculations in realistic cases.

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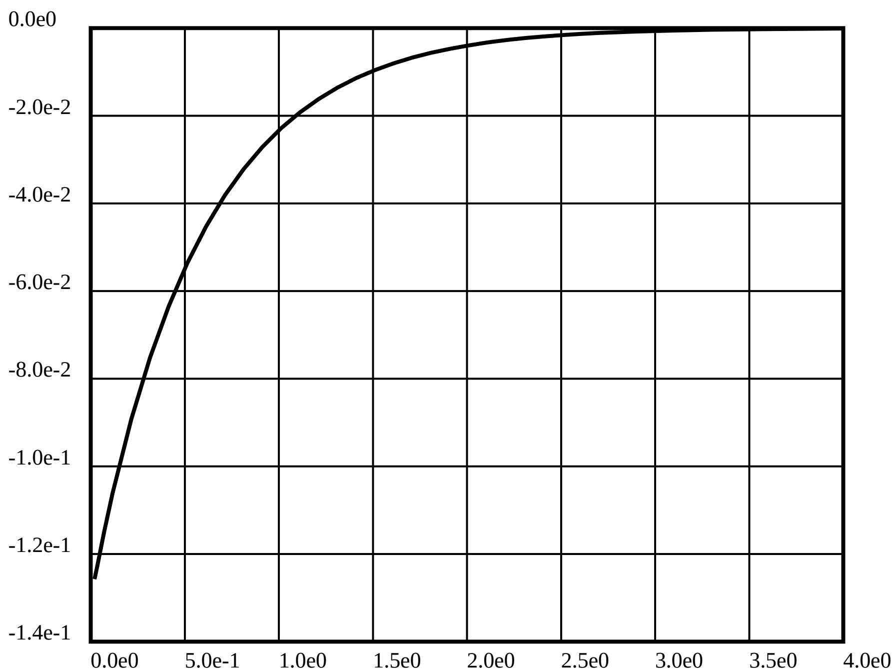

Figure 1: The Casimir energy of the massive scalar field on a unit interval as a function of the mass of the field.

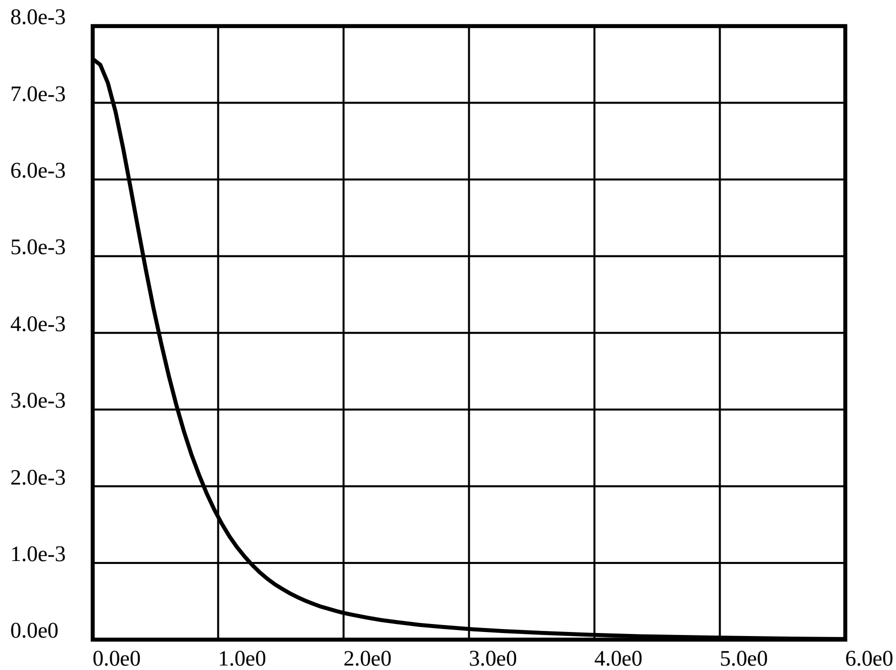

Figure 2: The Casimir energy of the scalar field inside of a spherical shell of radius 1 obeying Dirichlet boundary conditions as a function of the mass of the field.

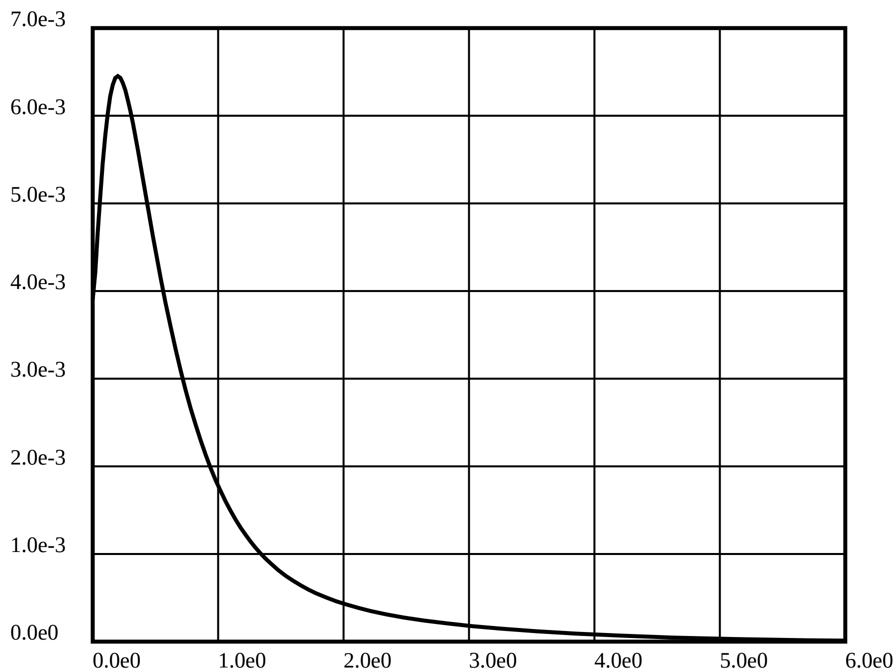

Figure 3: The Casimir energy of a massive scalar field in the whole space with Dirichlet boundary condition on a sphere of radius 1 as a function of the mass of the field.