Soliton production in high-energy collisions: a toy model

D.G. Levkov[a,b]*, S.M. Sibiryakov[a]†

[a]Institute for Nuclear Research of the Russian Academy of Sciences,
60th October Anniversary Prospect 7a, Moscow, 117312, Russia
[b]Moscow State University, Department of Physics,
Vorobjevy Gory, Moscow, 119899, Russia

Abstract

We consider a process of soliton production in collisions of a few highly energetic particles. This process is studied semiclassically in a simple field theoretical model. We observe that the properties of the process are qualitatively different in the cases when the energy of colliding particles is lower and higher than some critical value E_c. Namely, at $E < E_c$ direct tunneling to the soliton sector takes place, whereas the relevant semiclassical configurations with energies higher than E_c correspond to jumps on top of the potential barrier separating the soliton from the vacuum. We argue that the processes of soliton production remains exponentially suppressed up to extremely high energies.

1 Introduction

There are many field theoretical models where one discovers solitons, localized solutions to the classical field equations. In a situation when the de

*Email: levkov@ms2.inr.ac.ru
†Email: sibir@ms2.inr.ac.ru

Broglie wavelength of a soliton is much smaller than its size, the soliton corresponds to a "classical" state of quantum theory. The question we address here is whether it is possible to create a classical soliton colliding a few highly energetic particles.

The process under discussion belongs to a class of collision–induced tunneling processes which have been extensively studied in literature. Generally, the processes of this kind are exponentially suppressed up to a very high energies of colliding particles. In particular, the semiclassical results of Refs. [1, 2] confirm the exponential suppression of the induced tunneling processes in the cases of false vacuum decay in scalar theory and baryon number violation in SU(2) Higgs theory. Furthermore, investigations of toy models [3, 4] and unitarity arguments [5] indicate that the process of induced tunneling should remain exponentially suppressed, even when the energy of collision tends to infinity. In accordance with general expectations, our results show that the process of induced creation of a soliton is exponentially suppressed up to a very high energies.

The model we consider describes free scalar field $\phi(t, \mathrm{x})$ living in (1+1) dimensions on a half–line $\mathrm{x} > 0$, with interactions localized at the boundary point $\mathrm{x} = 0$. The action of the model is

$$S = \frac{1}{2}\int dt \int\limits_0^\infty d\mathrm{x}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - \frac{\mu}{g^2}\int dt\left[1 - \cos(g\phi(t,0))\right]. \qquad (1)$$

The second term represents boundary interaction with characteristic energy scale μ. The bulk mass m is introduced as an infrared regulator, it is supposed to be small compared to the boundary scale μ. In the main body of the article we adopt the limit $m \to 0$, as the small mass turns out to be irrelevant for our study.

The model (1) possesses a static soluton which, up to corrections of order m/μ, has the form

$$\phi_{\mathrm{sol}}(\mathrm{x}) = \frac{2\pi}{g}\mathrm{e}^{-m\mathrm{x}} .$$

This solution is localized near the boundary, $\mathrm{x} = 0$, so it is natural to call it "boundary soliton". The mass of the soliton,

$$M = \frac{2\pi^2}{g^2}m , \qquad (2)$$

is relatively small, as it is proportional to the bulk mass m.

The process we consider in this paper is creation of the soliton in collision of one or several particles with the boundary; the total energy of particles is assumed to be much larger than the soliton mass. To start with, let us consider the classical analog of this process, which is the creation of the soliton in collision of a classical wave packet with the boundary. It is easy to see that the classical process is possible only if the energy of the wave packet exceeds some threshold energy E_S. Indeed, the boundary value $\phi(t,0)$ of the relevant classical solution changes from 0 to $2\pi/g$ during the process. The classical solution passes the maximum of the boundary potential, π/g, at some moment of time, and thus the total energy of this solution is larger than the maximum boundary energy

$$E_S = \frac{2\mu}{g^2} \ . \tag{3}$$

One concludes that any state containing boundary soliton is separated from the vacuum by a potential barrier. It is straightforward to find static configuration "sitting" on top of the barrier, which we call "sphaleron" following Ref. [6]. This is an unstable static solution of the classical field equations representing the saddle point of static energy functional. In our model it has the same exponential form as the soliton but with the boundary value on top of the boundary potential,

$$\phi_S = \frac{\pi}{g} \mathrm{e}^{-m\mathrm{x}} \ .$$

The energy of the sphaleron is given by Eq. (3), again up to corrections of order m/μ.

We see that soliton production in collisions of particle(s) with the boundary is classically forbidden and hence exponentially suppressed at energies smaller than the sphaleron energy. The question is what happens when the energy grows. In this paper we study this question applying semiclassical methods. The semiclassical approximation is justified by the following observation. After rescaling of the field, $\phi \to \phi/g$, the coupling constant g enters the action only through the overall multiplicative factor $1/g^2$. Therefore, g^2 plays the role of the Planck constant $\hbar$, and the weak coupling limit corresponds to the semiclassical situation. This is the case we consider in this paper.

2 The boundary value problem

2.1 General formalism

In this work we adopt the semiclassical method of Ref. [7]. Let us consider the inclusive probability of tunneling from multiparticle states with energy E and number of particles N:

$$\mathcal{P}(E,N) = \sum_{i,f} \left| \langle f|\hat{\mathcal{S}}\hat{P}_E\hat{P}_N|i\rangle \right|^2 , \tag{4}$$

where $\hat{\mathcal{S}}$ is the S-matrix while $\hat{P}_E$ and $\hat{P}_N$ are projectors onto states with given energy and number of particles. The states $|i\rangle$ and $|f\rangle$ are perturbative excitations above the vacuum and soliton respectively. The function (4) can be calculated with the use of semiclassical methods, provided that the energy and initial number of particles are semiclassically large, $E = \tilde{E}/g^2$, $N = \tilde{N}/g^2$. The result has the exponential form,

$$\mathcal{P}(E,N) \propto \mathrm{e}^{-F(\tilde{E},\tilde{N})/g^2} .$$

Then, the exponent $F(\tilde{E})$ in a few–particle case is obtained as a limit of the multiparticle one:

$$F(\tilde{E}) = \lim_{\tilde{N}\to 0} F(\tilde{E},\tilde{N}) . \tag{5}$$

This method has been confirmed by calculations made in several models [8, 9, 10]. To simplify notations, we omit tilde over E and N below.

Transitions with large fixed initial number of particles are described by solutions of the so–called T/θ boundary value problem [7]. The latter is formulated on the contour ABCD in complex time shown in Fig. 1. Namely, the configurations describing tunneling should satisfy the classical field equations in the internal points of the contour,

$$(\partial_t^2 - \partial_{\mathrm{x}}^2 + m^2)\phi = 0, \quad \mathrm{x} > 0 , \tag{6a}$$

$$\partial_{\mathrm{x}}\phi = \mu \sin\phi, \quad \mathrm{x} = 0 . \tag{6b}$$

The Euclidean part BC of the contour may be interpreted as representing the evolution of the field under the barrier, "duration" T of this evolution is a parameter of solution. Field equations (6a), (6b) are supplemented by initial and final boundary conditions in the parts A and D of the contour

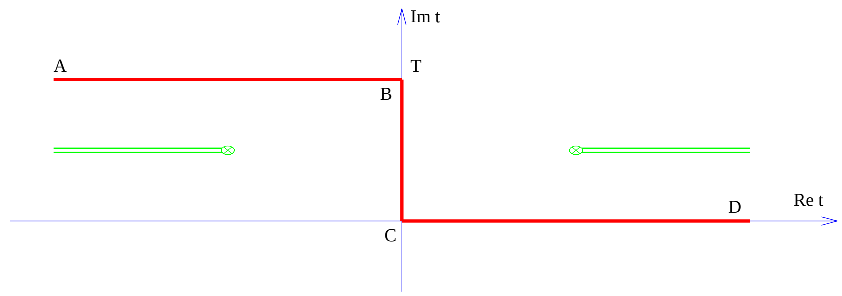

Figure 1: Contour in complex time where the boundary value problem is formulated

respectively. Namely, the field ϕ is real as $t \to +\infty$; it describes motion in the final state,

$$\mathrm{Im}\,\phi \to 0 \;\; \text{as} \;\; t \to +\infty \; . \tag{6c}$$

Evolution in the initial state is linear,

$$\phi(t' + iT, \mathrm{x})\Big|_{t' \to -\infty} = \frac{1}{(2\pi)^{1/2}} \int \frac{dk}{\sqrt{2\omega_k}} \left(f_k \mathrm{e}^{-i\omega_k t' + ik\mathrm{x}} + g_k^* \mathrm{e}^{i\omega_k t' - ik\mathrm{x}} \right) \; , \tag{6d}$$

and the boundary conditions in the part A of the contour relate positive and negative frequency components of the solution,

$$f_k = \mathrm{e}^{-\theta} g_k \; . \tag{6e}$$

The boundary condition (6e) can be understood as follows. In the limit $\theta \to +\infty$ it coincides with the Feynman boundary condition and thus corresponds to the initial state with semiclassically small number of particles. Finite θ represents the most favourable state with non–zero N. Given the values of T and θ, one determines complex solution $\phi(t, \mathrm{x}; T, \theta)$ from equations (6a)–(6e). Energy and initial number of particles for this solution are given by the familiar formulae:

$$E = \int dk \; \omega_k f_k g_k^*, \quad N = \int dk \; f_k g_k^* \; . \tag{7}$$

Alternatively, they can be determined by differentiating the action functional evaluated on the solution with respect to parameters T and θ,

$$E = \frac{\partial}{\partial T} \, \mathrm{Im} S(T, \theta) \; , \quad N = 2 \frac{\partial}{\partial \theta} \, \mathrm{Im} S(T, \theta) \; , \tag{8}$$

where S is the action of the model calculated along the contour $ABCD$. The suppression exponent of the process is given by the Legendre transform of the action functional,

$$F(E,N) = 2\,\mathrm{Im}S - N\theta - 2ET\;. \tag{9}$$

Below we refer to the problem (6) as "T/θ-problem", and use the term "θ–instanton" for the relevant semiclassical solution.

Two remarks are in order. First, the boundary value problem (6) does not guarantee that its solutions interpolate between states with and without the soliton. In the next subsection we describe additional requirements that should be imposed to ensure that solutions are relevant for tunneling.

Second, one observes that the solution $\phi(t,\mathrm{x})$ can be analytically continued to the complex time plane, and in this way the contour $ABCD$ may be deformed without affecting the integral (9) for the suppression exponent. The only thing one should worry about while deforming the contour is to avoid the singularities of solution, shown schematically by double lines in Fig. 1. Below it will be convenient not to be attached to a contour of any particular form. Instead, we look for solution $\phi(t,\mathrm{x})$ satisfying Eqs. (6a), (6b) in the entire complex time plane, with boundary conditions (6c) and (6e) imposed in the asymptotic regions D and A of the complex plane. When using this approach, one should guarantee, however, that the asymptotic regions A and D can be connected by a contour avoiding the singularities of solution.

2.2 Reformulation of the problem in complex plane

In this subsection we adapt the T/θ-problem (6) for the specifics of the model (1). The key observation here is that the characteristic frequency of solutions under consideration is of order of the boundary scale $\mu \gg m$. Thus, the mass term in the bulk equation (6a) can be neglected, and the general solution has the form

$$\phi(t,\mathrm{x}) = \phi_i(t+\mathrm{x}) + \phi_f(t-\mathrm{x}), \tag{10}$$

where ϕ_i and ϕ_f are the incoming and outgoing wave packets, respectively. They are related by boundary condition (6b):

$$\phi_i'(z) - \phi_f'(z) = \mu \sin\left(\phi_i(z) + \phi_f(z)\right), \tag{11}$$

where we have promoted t to the complex variable z. It is natural to consider ϕ_i and ϕ_f as analytic functions of z, and reformulate the rest of the problem (6), i.e. conditions (6c) and (6e), in the complex z–plane.

Let us consider asymptotic future, $t \to +\infty$ (region D in Fig 1). In this limit field $\phi(t, \mathrm{x})$ is represented by the outgoing wave packet $\phi_f(t - \mathrm{x})$ whose argument $z = t - \mathrm{x}$ runs all the way along the real axis as x changes from 0 to $+\infty$. So, condition (6c) can be written in the following way,

$$\mathrm{Im}\ \phi_f(z) = 0, \quad \text{when} \quad z \in \mathbb{R}\ . \tag{12}$$

On the other hand, it is the in–going wave packet $\phi_i(t + \mathrm{x})$ which survives in the asymptotic past $t \to -\infty + iT$ (region A of Fig. 1), the argument of function $\phi_i(z)$ runs along the line Im $z = T$ when $\mathrm{x} \in [0, +\infty)$. So, condition (6e) is reformulated in terms of function ϕ_i. Namely, one performs Fourier expansion of ϕ_i along the line Im $z = T$,

$$\phi_i(z) = \int dk\ \phi_i(k) \mathrm{e}^{ik(z-iT)} = \\ = \int\limits_{k>0} dk\ \left\{ \phi_i(k) \mathrm{e}^{ik(z-iT)} + \phi_i(-k) \mathrm{e}^{-ik(z-iT)} \right\} . \tag{13}$$

Taking into account that $z = t' + iT + \mathrm{x}$ for the initial wave packet, one compares Eq. (13) to Eq. (6d) and finds that positive and negative frequency components f_{-k} and g^*_{-k} of solution are proportional to $\phi_i(-k)$ and $\phi_i(k)$, $k > 0$, respectively. Equation (6e) takes the form

$$\phi_i(-k) = \mathrm{e}^{-\theta} \phi_i{}^*(k)\ , \quad k > 0. \tag{14}$$

Given this condition, function ϕ_i can be represented as

$$\phi_i(z) = \chi(z - iT) + \mathrm{e}^{-\theta}[\chi(z^* + iT)]^*, \tag{15}$$

where function

$$\chi(z) = \int_0^\infty dk\ \phi_i(k) \mathrm{e}^{ikz}\ . \tag{16}$$

is regular in the upper half plane of its complex argument. Equation (15) provides an alternative formulation of the θ–boundary condition. Note that if the number of incoming particles is finite, the in-going wavepacket is localized in space. This implies

$$\phi_i(z) \to 0\ , \quad z \to \pm\infty + iT\ . \tag{17}$$

To summarize, the T/θ–problem is represented by equations (11), (12) and (15) formulated in the complex z–plane.

To make sure that solution of the above problem is relevant for tunneling, one should check that the value of the field at the boundary $\mathrm{x}=0$ has correct asymptotics in the beginning and the end of the process. Namely, in the case of direct tunneling with soliton in the final state, $\phi(t,0)$ must change from 0 to 2π. In Sec. 4 we encounter configurations describing creation of sphaleron at $t \to +\infty$; in that case $\phi(t,0)$ changes from 0 to π. Thus, one obtains the following conditions:

$$\phi_i(z) + \phi_f(z) \to 0 \qquad \text{as} \;\; z \to -\infty + iT, \tag{18a}$$

$$\phi_i(z) + \phi_f(z) \to 2\pi \;\; \text{or} \;\; \pi \qquad \text{as} \;\; z \to +\infty. \tag{18b}$$

Finally, let us analyze the issue of existence of a time contour connecting asymptotic regions A and D in Fig. 1. To this end, note that any singularity z_i^s of the function $\phi_i(z)$ produces a whole half–line of singularities $t^s = z_i^s - \mathrm{x}$ in the time plane, which starts from the point z_i^s and extends to the left parallel to the real time axis, see Fig. 1. One requires that all these half–lines of singularities of the function $\phi_i(t+\mathrm{x})$ in the strip $\mathrm{Im}\, t \in [0,T]$, in complex time plane, should be located to the left of the relevant contour. Analogously, one observes that all the singularities of function $\phi_f(t-\mathrm{x})$ in this strip should be located to the right of the relevant contour. It is clear that one is always able to find some contour connecting regions A and D which leaves the singularities of $\phi_i(t+\mathrm{x})$ and $\phi_f(t-\mathrm{x})$ to the left and right of it respectively, provided that the singularities of these two functions do not coincide. In terms of the variable z this amounts to requiring that the singularities, inside the strip $\mathrm{Im}\; z \in [0,T]$, of the initial and final wave packets $\phi_i(z)$ and $\phi_f(z)$ are situated at different points. Note that this condition is non-trivial, as the functions ϕ_i and ϕ_f are related by differential equation (11).

Formula (1) for the action can also be rewritten in terms of wave packets:

$$S = \int_{\mathcal{C}} dz \left\{ \frac{1}{2}(\phi_i + \phi_f)(\phi_i' - \phi_f') - \mu(1 - \cos(\phi_i + \phi_f)) \right\}. \tag{19}$$

The form of the coutour $\mathcal{C}$ here is somewhat similar to the time contour ABCD in Fig. 1, it interpolates between the asymptotic regions $z \to -\infty + iT$ and $z \to +\infty$, leaving the singularities of functions ϕ_i and ϕ_f in the strip $\mathrm{Im}\, z \in [0,T]$ to the left and right respectively.

3 Direct soliton production at low energies

We now proceed to solutions with $\theta \neq 0$. Let us make the following observation: if ϕ_i is *real* on the real axis, function ϕ_f determined from Eq. (11) with real initial condition, is automatically real at the real axis. Thus, all we need is an Ansatz for ϕ_i, which satisfies condition (15) and is real on the real axis. One constructs the required Ansatz in the following way[1]:

$$\phi_i = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\theta|n|} \left(i \ln \left[\frac{\mu}{2}(z - iT_0 - i2Tn) \right] - i \ln \left[\frac{\mu}{2}(z + iT_0 - i2Tn) \right] \right) . \quad (20)$$

Here T_0 is a real parameter. Is straightforward to check that function ϕ_i determined by (20) can be represented in the form (15).

In order to determine the parameter T_0 of the Ansatz, let us analyze equation (11) in the vicinity of the point $z = iT_0$. One represents function ϕ_i in the form

$$\phi_i = i \ln \left[\frac{\mu}{2}(z - iT_0) \right] + R_i(z) , \quad (21)$$

where $R_i(z)$ is regular at $z = iT_0$. Two leading terms of the power series expansion of Eq. (11) at the point $z = iT_0$ yield,

$$\mathrm{e}^{i(R_i(iT_0)+\phi_f(iT_0))} = -1 , \quad (22)$$

$$R_i'(iT_0) = 0 . \quad (23)$$

These formulae deserve a comment. Considering Eq. (11) with given ϕ_i as an ordinary differential equation for function ϕ_f, one might expect all Taylor coefficients of ϕ_f to be determined in terms of function $R_i(z)$ and free integration constant $\phi_f(iT_0)$. However the fact that $z = iT_0$ is a singular point of Eq. (11) makes situation quite different. Requirement of regularity of ϕ_f at this point fixes the value of $\phi_f(iT_0)$ according to Eq. (22), while the role of integration constant is played by $\phi_f'(iT_0)$; besides, constraint (23) on function R_i appears. The latter constraint enables to determine parameter T_0,

$$T_0 = T\alpha(\theta) , \quad (24)$$

[1]In fact, one can show that (20) is the most general Ansatz for function ϕ_i, once requirement of its reality on the real axis is imposed.

where function $\alpha(\theta)$ is implicitly defined by relation

$$2\alpha^2 \sum_{n=1}^{\infty} \frac{e^{-\theta n}}{n^2 - \alpha^2} = 1 \ . \qquad (25)$$

Function ϕ_i is now completely fixed, and ϕ_f can be obtained by numerical integration of Eq. (11). This issue is discussed at the end of this subsection.

Let us evaluate the imaginary part of action on the tunneling solution. Surprisingly, the detailed knowledge of ϕ_f is not needed for this purpose. Reality of solution on the real axis implies that the complex conjugate action S^* is given by an integral of the same function as in Eq. (19), but with different contour of integration $\mathcal{C}^*$, which is complex conjugate to $\mathcal{C}$. Thus,

$$2\,\mathrm{Im}\,S = -i(S - S^*) = -i\left(\int_{\mathcal{C}} \mathcal{L}dz - \int_{\mathcal{C}^*} \mathcal{L}dz\right) = -i \oint_{\mathcal{C}_o} \mathcal{L}dz \ . \qquad (26)$$

In the last equality we deformed the sum of the contours $\mathcal{C}$ and $\mathcal{C}^*$ into contour $\mathcal{C}_o$ enclosing singularities[2] $z = \pm iT_0$ of the function ϕ_i. Calculation of integral (26) is straightforward:

$$2\,\mathrm{Im}\,S = 4\pi\,\mathrm{Im}\,\phi_f(iT_0) + 4\pi \ . \qquad (27)$$

Using Eq. (22) one substitutes $R_i(iT_0)$ for $\phi_f(iT_0)$, takes the former from Eq. (20) and performs summation. The result is

$$2\,\mathrm{Im}\,S = 4\pi \ln(\mu T \alpha) + 4\pi - 16\pi \int_0^\infty \frac{\mathrm{sh}^2\left(\frac{\alpha y}{2}\right)}{e^{y+\theta} - 1} \frac{dy}{y} \ . \qquad (28)$$

Energy and number of incoming particles are determined from expression (28) in the standard way, see Eqs. (8). We find that energy is given by the formula

$$E = \frac{2\pi}{T} \ , \qquad (29)$$

while the number of incoming particles is

$$N = 4\pi \int_0^\infty \frac{\mathrm{sh}^2\left(\frac{y\alpha(\theta)}{2}\right)}{\mathrm{sh}^2\left(\frac{y+\theta}{2}\right)} \frac{dy}{y} \ . \qquad (30)$$

[2]In the course of deformation the contour does not cross singularities of ϕ_f according to conditions discussed in Sec. 2.2.

In the limit $\theta \to +\infty$ corresponding to the case of a few incoming particles, the formula for the suppression exponent simplifies,

$$F\Big|_{N=0} = 4\pi \ln \left[\frac{2\pi\mu}{E}\right] . \tag{31}$$

At first sight Eq. (31) suggests that suppression vanishes when energy reaches the value $2\pi\mu$. In fact, this is not the case: formula (31) is inapplicable at energies above some critical energy $E_c < 2\pi\mu$. The point is that no solution $\phi_f(z)$ of Eq. (11) with required properties exists at energies $E > E_c$. Let us clarify this issue.

While the analysis can be carried for arbitrary θ, it is particularly transparent in the case $\theta = +\infty$. In this limit the Ansatz (20) simplifies:

$$\phi_i = i \ln \frac{z - iT}{z + iT} . \tag{32}$$

It is convenient to consider Eq. (11) on the real axis. Introducing $u = \phi_i + \phi_f$ and $\zeta = z/T$, one writes Eq. (11) in the following form,

$$\frac{du}{d\zeta} = -\lambda \sin u - \frac{4}{\zeta^2 + 1} , \tag{33}$$

where $\lambda = \mu T$. In new terms requirements (18), stating that solution is relevant for soliton production, imply the following boundary conditions for u along the real axis:

$$u \to 2\pi \ , \quad \zeta \to -\infty \ , \tag{34a}$$

$$u \to 2\pi \quad \text{or} \quad \pi, \quad \zeta \to +\infty \ . \tag{34b}$$

Condition (34a) follows from Eq. (18a) when one takes into account that asymptotic region $z \to -\infty + iT$ and the real axis lie on different sides of logarithmic cut of function ϕ_i. Condition (34a) fixes solution of equation (33) uniquely. Then, the question is whether this solution satisfies boundary condition (34b). We have analyzed this issue numerically. The result is as follows. When λ is greater than λ_c, the solution of Eqs. (33), (34a) has the correct asymptotics (34b), so, the configuration does describe production of the soliton. On the other hand, when one lowers λ below the critical value λ_c, the asymptotics of the function u at $\zeta \to +\infty$ change to 0. As u represents the value of the field at the boundary x = 0 (c.f. Eq. (10)), one concludes that the final state of the process at $\lambda < \lambda_c$ is the trivial vacuum, and no actual tunneling takes place. At the critical value, $\lambda = \lambda_c$, function u tends to π when $\zeta \to +\infty$, and solution describes production of the sphaleron in the final state. Numerically, we find

$$\lambda_c = 2.62 \; , \tag{35}$$

this value corresponds to energy $E_c = 1.2 E_S$. Tunneling at critical energy is still exponentially suppressed:

$$F(N = 0, E = E_c) = 4\pi \ln \lambda_c \; . \tag{36}$$

The picture we encounter here has been observed recently in quantum mechanics of two degrees of freedom [10] and in gauge theory [2]. Results obtained in both cases indicate that transitions at energies higher than critical proceed in two stages, creation of sphaleron and its subsequent quantum decay into relevant final state. Probability of the latter process is of order one, while the former is exponentially suppressed. Corresponding semiclassical solutions contain sphaleron at $t \to +\infty$. In order to find such solutions one has to abandon the requirement of reality of the incoming wave packet ϕ_i on the real axis and thus the Ansatz (20).

4 Jumps onto sphaleron

At energies higher than E_c the Ansatz (20) is no longer applicable. Still, one can extract some information on the properties of ϕ_i from analysis of Eqs. (11), (15). First, one can show that the singularity of ϕ_i situated inside the strip $\mathrm{Im}\, z \in [0, T]$ is necessarily logarithmic. Let us denote the position of this singularity by iT_0. The function ϕ_f is regular at the point $z = iT_0$, so ϕ_f and ϕ_f' can be replaced by constants in the small vicinity of this point. Then, integration of Eq. (11) produces function $\phi_i(z)$ with logarithmic singularity.

Thus the function ϕ_i has the form (21) with the function R_i regular in the strip $\mathrm{Im}\, z \in [0, T]$. Besides, one observes that conditions (22), (23) on R_i are still valid, as their derivation does not make use of any particular Ansatz for ϕ_i. Finally, due to Eq. (15), presence of the logarithmic singularity of ϕ_i at $z = iT_0$ entails existence of another singularity of ϕ_i with the structure

$$-ie^{-\theta} \ln \left[\frac{\mu}{2} (z - i(2T - T_0)) \right] \; . \tag{37}$$

This information enables one to cast the T/θ problem in the limit $\theta \to +\infty$ into the form directly accessible to numerical analysis. We now proceed to formulation of the corresponding equations.

According to Eq. (15), the function ϕ_i becomes regular in the half-plane $\mathrm{Im}\, z > T$ when $e^{-\theta}$ tends to zero. As for its singularity in the strip $0 < \mathrm{Im}\, z < T$, it hits the line $\mathrm{Im}\, z = T$ in the considered limit. To demonstrate this explicitly, we write

$$\phi_i = i \ln \left[\frac{\mu}{2}(z - iT_0) \right] - ie^{-\theta} \ln \left[\frac{\mu}{2}(z - i(2T - T_0)) \right] + \tilde{R}_i(z) \;, \tag{38}$$

where $\tilde{R}_i(z)$ is regular both at iT_0 and $i(2T - T_0)$. Equation (23) then implies

$$\frac{e^{-\theta}}{T - T_0} = -2\tilde{R}_i{}'(iT_0) \;. \tag{39}$$

This equation implies that T_0 approaches T when $e^{-\theta} \to 0$. We conclude that after taking the limit $\theta \to +\infty$ the function ϕ_i has the form

$$\phi_i = i \ln \left[\frac{\mu}{2}(z - iT) \right] + \tilde{R}_i(z) \;, \tag{40}$$

where $\tilde{R}_i(z)$ is regular in the upper half-plane.

As in Sec. 3, it is convenient to consider equations for functions ϕ_i and ϕ_f on the real axis[3] $z = x \in \mathbb{R}$. Again, one should be careful about the asymptotics $x \to \pm\infty$. The function ϕ_i satisfies condition (17), as we are dealing now with the case of finite number of incoming particles. Taking into account the logarithmic cut of ϕ_i in the strip $0 < \mathrm{Im}\, z < T$ (c.f. Sec. 3) and conditions (18), one obtains

$$\phi_i \to 2\pi, \quad \phi_f \to 0 \qquad \text{when} \quad x \to -\infty \;, \tag{41a}$$

$$\phi_i \to 0, \quad \phi_f \to \pi \qquad \text{when} \quad x \to +\infty \;. \tag{41b}$$

Note that asymptotics of ϕ_f at $x = +\infty$ reflects formation of the sphaleron in the end of the tunneling process.

[3] Do not confuse x which is the real part of the variable z, with the spatial coordinate x.

A convenient expression for the action functional is obtained in the following way:

$$2\,\mathrm{Im}\, S = 2\,\mathrm{Im}\int_{\mathcal{C}} dz(\phi_f\phi_i' - \mu(1-\cos(\phi_i+\phi_f)))$$

$$= 4\pi\,\mathrm{Im}\,\tilde{R}_i(iT) + 4\pi + \tag{42}$$

$$+\, 2\,\mathrm{Im}\int_{-\infty}^{\infty} dx(\phi_f\phi_i' - \mu(1-\cos(\phi_i+\phi_f)))\ . \tag{43}$$

In the second line we used Eq. (22) and took the limit $T_0 \to T$. Integration in the last term is performed along the real axis; the first two terms account for the residue of the integrand in the logarithmic singularity of ϕ_i. Finally, let us present convenient formulae for the energy of solution. One finds two different expressons for the energies of initial and final state; we denote these energies E_i and E_f respectively[4]. For the final energy it is straightforward to obtain

$$E_f = 2\mu + \int_{-\infty}^{\infty} dz(\phi_f')^2\ , \tag{44}$$

where the first term is due to presence of the sphaleron in the final wave packet. The analogous expression for initial energy has the form,

$$E_i = \int_{-\infty+iT}^{+\infty+iT} dz(\phi_i')^2\ . \tag{45}$$

In the limit $e^{-\theta} \ll 1$ integral in (45) is saturated by contribution of the singularity at $z = i(2T - T_0)$. One obtains

$$E_i = 2\pi\frac{e^{-\theta}}{T-T_0} = -4\pi\tilde{R}_i'(iT)\ , \tag{46}$$

where in the second equality we used Eq. (39).

We have performed numerical solution of the following set of equations formulated on the real axis: differential Eq. (11) with boundary conditions (41), condition of reality of ϕ_f (Eq. (12)), analyticity condition (40). Details of our numerical method are presented elsewhere [11]. Let us mention here

[4] Evidently, E_i equals E_f for any solution of equations of motion.

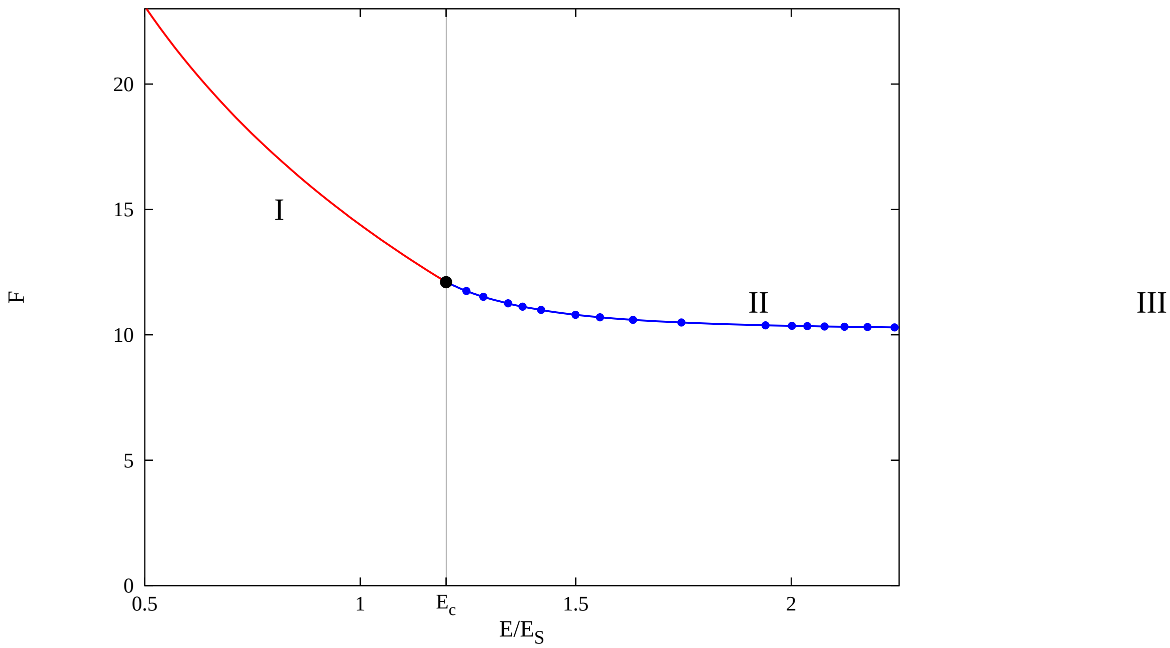

Figure 2: Dependence of the suppression exponent F on the collision energy E, measured in units of sphaleron energy $E_S = 2\mu/g^2$. Tunneling process is driven by physically different mechanisms in parts I and II of the graph. Points represent numerical data obtained in region II.

an interesting property of the solutions. One notes that the above set of equations is invariant under the transformation

$$\phi_i(z) \mapsto 2\pi - \big(\phi_i(-z^*)\big)^* \,, \tag{47a}$$

$$\phi_f(z) \mapsto \pi - \big(\phi_f(-z^*)\big)^* \,. \tag{47b}$$

We find that tunneling solutions at $E > E_c$ are symmetric with respect to this transformation. This property reflects that they are qualitatively different from solutions at energies lower than critical.

The imaginary part of action, energy and suppression exponent are calculated according to formulae (43), (44), (46); we use equality of initial and final energies as a cross–check of precision of numerical calculations. Results for suppression exponent $F(N = 0)$ are presented in Fig. 2. They cover the interval $E_c < E < 2.3E_S$ of the region II. Let us stress that our numerical solutions describe tunneling with exactly *zero* number of incoming particles. The fact that one is able to find such solutions is a peculiarity of the model. Unlike in the case of more complicated systems [2] we do not need to perform calculations at finite N.

5 Conclusions

Let us summarize the results obtained in this article. We calculated semiclassically the probability $\mathcal{P}$ of the soliton production in collision of one or several particles. In the leading semiclassical approximation it has the exponential form,

$$\mathcal{P} \propto \mathrm{e}^{-F/g^2}.$$

Our results for the dependence of the suppression exponent F on energy of incoming particle(s) are collected in Fig. 2. We observe that if the collision energy is smaller than critical value $E_c = 1.2E_S$ (region I of the graph), tunneling occurs in a conventional way with the relevant semiclassical configurations ending up directly in the soliton sector. We find a simple analytic formula for the suppression of such transitions:

$$F(E) = 4\pi \ln \left[\frac{\pi E_S}{E} \right] , \quad E < E_c \; . \tag{48}$$

Formula (48), if continued to the energies higher than the critical energy E_c, would show that the transitions become unsuppressed at energy πE_S. Hovewer, it is incorrect at $E > E_c$, as the solutions describing direct tunneling cease to exist at energies higher than the critical one.

Semiclassical configurations with energies above E_c are obtained numerically. We find that their symmetry properties are different from the ones below the critical energy, and, what is more important, the tunneling mechanism they describe is entirely different. Instead of tunneling directly to the other side of the barrier, the system jumps on its top, thus creating the sphaleron configuration which then decays producing the soliton in the final state. The second stage of the process, decay of the sphaleron into the soliton, proceeds with probability of order one. Still, transitions remain exponentially suppressed due to considerable rearrangement the system has to undergo during the first stage of the process, i.e. formation of the sphaleron.

Finally, let us speculate on exponential suppression of soliton production at higher energies. From Eqs. (8), (9) one obtains:

$$\frac{\partial F}{\partial E} = -2T \; . \tag{49}$$

Let us also recall that the value of T decreases with increase of energy. The highest–energy solution which we managed to obtain numerically has $E = 2.3E_S$, $T = 0.03$, $F = 10.3$. Thus, from Eq. (49) we obtain:

$$\left|\frac{\partial F}{\partial E}\right| < 0.06 \quad \text{at} \quad E > 2.3E_S \; .$$

Integrating this inequality, we learn that the process under discussion remains exponentially suppressed at least up to 170 sphaleron energies.

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