On correlation functions in perturbed minimal models

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October 18, 2004

Abstract

We study two-point correlation functions of spin operators in the minimal models $\mathcal{M}_{p,p'}$ perturbed by the field Φ_{13} using both the framework of conformal perturbation theory [1] and the form-factor approach [2]. This article is a review of the results in [3].

1 Introduction

There are several reasons to study minimal models of CFT and their perturbations.

1. Minimal models correspond to the fixed points of the Wilson RG in two dimensions. They describe different types of critical behavior in statistical mechanics and different types of UV behavior in QFT.

2. The minimal models dressed by the Liouville gravity are a very interesting example of string theory.

3. Relevant perturbations of minimal models describe the neighborhood of the critical point in statistical mechanics and correspond to super-renormalizable models in QFT.

4. The minimal models and their Φ_{13} perturbations are exactly solvable.

These models [4] have the infinite-dimensional conformal symmetry $Vir \times Vir$, where Vir is the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m} \quad (1)$$

with $c = 1 - \frac{6(p-q)^2}{pq}$, where p and q are coprime positive integers. The algebra of the local fields consists of a finite number of irreducible representations of the conformal algebra, and the corresponding primary fields Φ_{nm} with $0 < n < p-1$ and $0 < m < q-1$ have the dimensions

$$\Delta_{nm} = \frac{(pm - pq)^2 - (p-q)^2}{4pq}. \quad (2)$$

All correlation functions in these theories are known and are expressed by the Feigin–Fucks–Dotsenko–Fateev integral representation [5].

A. B. Zamolodchikov [6] considered the perturbed models $M_{pp'}$,

$$A_{pp'} = M_{pp'} + g\int \phi(y)\, d^2y, \quad (3)$$

where $\phi = \Phi_{13}$. Because the dimension $2\Delta_{13}$ is less than the dimension of the space–time, this perturbation is relevant and describes the scaling neighborhood of the critical point. The theory $A_{pp'}$ is renormalizable (more precisely, superrenormalizable). This means that UV divergences in the correlation functions, which are formally defined in the perturbed theory as

$$\langle A_1^0(x_1)\cdots A_N^0(x_N)\rangle \stackrel{def}{=} \langle A_1^0(x_1)...A_N^0(x_N)\exp(-g\int \phi(y)d^2y\rangle_{CFT} = \\ = \sum_{n=0}^{\infty}\frac{(-g)^n}{n!}\int \langle A_1^0(x_1)\cdots A_N^0(x_N)\phi(y_1)\cdots\phi(y_n)\rangle_{CFT} d^2y_1\cdots d^2y_n, \quad (4)$$

can be deleted if the new basis of fields A_j is taken instead of the old one A_j^0. The renormalizable fields A_j are in one-to-one correspondence with the fields from the nonperturbed model (they have the same notation without the upper index 0) and are distinguished from them by adding a finite number of counterterms. These counterterms are the fields A_k^0 with dimensions less than the dimension of the field A_j; the coefficients before them depend on the UV cutoff ϵ and are fixed by the requirement that correlation functions should be finite. More precisely, this procedure fixes only the infinite part of these counterterms

$$A_j(x) = A_j^0(x) + \sum_{k \neq j} \epsilon^{2(\Delta_k - \Delta_j)} U_j^k(g,\epsilon) A_k^0(x),$$

where $U_j^k = \sum_N U_j^k(N)[g\epsilon^{2(1-\Delta)}]^N$ and the sum ranges N such that $N(1-\Delta) < (\Delta_j - \Delta_k)$. The simplest way to fix finite counterterms is executed in the formula above: the diagonal coefficient should be taken equal to 1 and the finite parts of the coefficients before the other fields equal to 0. This "minimal" procedure for fixing counterterms is equivalent to the requirement that the renormalized fields be eigenvectors of the dilatation operator, i.e., the fields with definite scaling dimensions, and have the same UV asymptotic behavior as the conformal one. As shown in [6], the models $A_{pp'}$ are integrable massive models with factorizable scattering. Below, we mainly consider the case of the perturbed minimal models $(p, p') = (2, 2n+1)$. It was shown in [7] that the corresponding perturbed models have n distinct particles whose scattering is diagonal.

Thus, both the UV and the on-shell properties are known in the models $A_{2,2n+1}$. A natural question arises: Can this information be used to compute correlation functions? There are two procedures for evaluating correlation functions:

1. The first is the short-distance expansion combining Conformal Perturbation Theory (CPT) [1] with knowledge of Vacuum Expectation Values (VEV) of local fields in the perturbed theory [8], [12].

2. The second is the long-distance expansion based on representing the correlation functions in the form of Spectral series and using the Form-Factor Bootstrap Approach [2].

In the following sections, we briefly expose the main ideas of these two approaches and compare them using the well-known g–m relation [9]

$$\pi g = -\frac{(\xi+1)^2}{(\xi-1)(2\xi-1)} \left(\gamma(\frac{3\xi}{\xi+1}) \gamma(\frac{\xi}{\xi+1}) \right)^{\frac{1}{2}} m^{\frac{4}{\xi+1}} . \qquad (5)$$

Here and hereafter, the mass parameter m is related to the mass of the lightest particle m_1 by

$$m = m_1 \left(\frac{\pi\gamma(1-\frac{\xi}{2})\gamma(\frac{1+\xi}{2})}{8\sin\pi\xi} \right)^{\frac{1}{2}},$$

$$\gamma(x) \;=\; \frac{\Gamma(x)}{\Gamma(1-x)},$$

and $\xi = \frac{p}{p'-p}$.

2 Conformal Perturbation Theory

As an example, we evaluate the two-point correlation function of the fields $\Psi(x)$, which denotes the local field Φ_{12}. Although the correlation function of the renormalized fields is free from UV divergences, they still have IR divergences, which arise in each term in the perturbation series from the integrals over y_k. The IR cutoff R, which can be considered a radius of the sphere in which our system is placed, should be introduced. If we computed all the integrals and summed, then in the limit $R \to \infty$, we would obtain a finite expression that depends nonanalytically on the coupling constant g. Because such a way is not usable, we use the CPT method [1] to evaluate the correlation functions. We consider the operator product expansion in our theory,

$$A_m(x)A_a(0) = \sum_b C^{A_b}_{A_m A_a}(r)A_b(0)\,. \tag{6}$$

The CPT idea is based on the hypothesis that the structure functions C^l_{km} are analytical in the coupling constant g if the basis in the space of fields consists of renormalizable fields A_j with definite scaling dimensions. This means that the structure functions are defined by series

$$C^l_{mk}(x) = |x|^{2(\Delta_l-\Delta_m-\Delta_k)} \sum_{N=0}^{\infty} C^l_{mk}(N)[g|x|^{2(1-\Delta)}]^N, \tag{7}$$

which converge for sufficiently small x because $\Delta < 1$. This hypothesis provide the way to evaluate $C^l_{mk}(x)$ by perturbation theory. Indeed, substituting (6) and (7) in the correlation functions

$$\langle A^0_p(\infty)A_m(x)A_k(0)\rangle_R = \langle A^0_p(\infty)A_m(x)A_k(0)e^{-g\int_{|y|<R} d^2y\phi(y)}\rangle_{CFT} \tag{8}$$

and expanding this expression in a perturbative series in the coupling constant g, we obtain the triangle system of linear equations for the coefficients $C_{mk}^l(N)$. Taking the vacuum average of both sides of the OPE for $\langle\Psi(x)\Psi(0)\rangle$, we obtain the double series for the two-point correlation function:

$$\langle\Psi(x)\Psi(0)\rangle = \sum_l C_{\psi,\psi}^l(x)\langle A_l(0)\rangle =$$
$$= \sum_l \sum_{N=0}^{\infty} |x|^{2(\Delta_l - 2\Delta_\psi)} C_{\psi,\psi}^l(N)[g|x|^{2(1-\Delta)}]^N \langle A_l(0)\rangle. \quad (9)$$

Both sums in the right-hand side are finite. The first sum (internal) converges, as mentioned above, at least for small $|x|$, when $g|x|^{2(1-\Delta)} < 1$ is also small because of the inequality $\Delta < 1$. The second sum (external) converges because the dimensions Δ_l are increasing. The convergence of these series allows restricting ourself to a small number of terms in the region of small x. The obtained expression depends on two types of data: on the coefficients $C_{mk}^l(N)$ (the procedure for calculating them is described above) and on the VEV $\langle A_l(0)\rangle$. From counting dimensions, we have

$$\langle A_l(0)\rangle = O_l(\xi) m^{2\Delta_l},$$

where the first factor $O_l(\xi)$ is independent of the coupling constant g and the second factor depends on g nonanalytically according to (5). Namely the VEV contains all nonperturbative information in itself.

The VEV of the primary local fields in the perturbed minimal model were evaluated in [12]. The Lukyanov and Zamolodchikov formula for the VEV [12] can be written as

$$\langle\Phi_{1k}\rangle = (-1)^{k-1} m^{2\Delta_{1k}} Q(1 - \xi(k-1)) \,. \quad (10)$$

The function $Q(\eta)$ is defined in terms of the following exponent of an integral, which is understood in the sense of analytic continuation from the region where it converges:

$$Q(\eta) = \exp \int_0^\infty \frac{dt}{t} \left(\frac{\cosh(2t)\sinh((\eta-1)t)\sinh((\eta+1)t)}{2\cosh(t)\sinh(\xi t)\sinh((\xi+1)t)} - \frac{\eta^2 - 1}{2\xi(\xi+1)} e^{-2(\xi+1)t} \right) \,. \quad (11)$$

The one-point VEV for the first nontrivial descendent operators are also known in the analytic form [8],

$$\langle L_{-2}\bar{L}_{-2}\Phi_{1k}\rangle = -(1+\xi)^4\mathcal{W}(1-\xi(k-1))m^4\langle\Phi_{1k}\rangle\,, \qquad (12)$$

where the function $\mathcal{W}(\eta)$ is

$$\mathcal{W}(\eta) := \frac{1}{\xi^2(\xi+1)^2}\gamma(\frac{1+\eta+\xi}{2})\gamma(\frac{\eta-\xi}{2})\gamma(\frac{1-\eta+\xi}{2})\gamma(-\frac{\eta+\xi}{2})\,.$$

The integral in (11) can be evaluated explicitly in terms of gamma-functions when the argument is rational [3].

3 Short-distance expansion

For simplicity, we restrict our attention to the case of the $M_{2,7}$ perturbed model (the $M_{2,5}$ case was considered in [1]). Retaining only the first leading terms, which were evaluated in [3], we obtain

$$\begin{aligned}\langle\Psi(x)\Psi(0)\rangle &= C^{I}_{\Psi\Psi}(r)\langle I\rangle + C^{\Phi}_{\Psi\Psi}(r)\langle\Phi(0)\rangle + C^{\Phi_{15}}_{\Psi\Psi}(r)\langle\Phi_{15}\rangle \\ &+ C^{L_{-2}\bar{L}_{-2}I}_{\Psi\Psi}(r)\langle L_{-2}\bar{L}_{-2}I\rangle + C^{L_{-2}\bar{L}_{-2}\Phi}_{\Psi\Psi}(r)\langle L_{-2}\bar{L}_{-2}\Phi(0)\rangle + \cdots\,. \end{aligned} \qquad (13)$$

Only zeroth-order terms in the structure functions $C^{L_{-2}\bar{L}_{-2}I}_{\Psi\Psi}(r)$ and $C^{L_{-2}\bar{L}_{-2}\Phi}_{\Psi\Psi}(r)$ should be retained and the structure functions $C^{\Phi_{15}}_{\Psi\Psi}(r)$ $C^{I}_{\Psi\Psi}(r)$ and $C^{\Phi}_{\Psi\Psi}(r)$ should be evaluated up to the first order to not dominate the accuracy restricted by knowing only the VEV of the first descendants. For the first-order corrections to the structure functions, we use the formula

$$\begin{aligned} C^{K\,(1)}_{\Psi\Psi}(r) := \lim_{R\to\infty}\Bigg[&-g\int_{|y|<R} < \tilde{\mathcal{A}}^K(\infty)\Phi(y)\Psi(x)\Psi(0) >_{CFT} d^2y \\ &+\pi g\sum_A \frac{C^A_{\Psi\Psi}C^K_{\Phi A}}{\Delta_K-\Delta_A-\Delta_\Phi+1}R^{2(\Delta_K-\Delta_A-\Delta_\Phi+1)}r^{2\Delta_A-4\Delta_\Psi}\Bigg]\,. \end{aligned} \qquad (14)$$

Using FFDF formulas for the four-point correlation functions we can obtain the following expressions for the first-order corrections to the structure functions :

$$C^{I(1)}_{\Psi\Psi}(r) = g\left(\frac{\gamma(\frac{\xi}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})}\frac{\gamma(\frac{2}{\xi+1})}{\gamma(\frac{2-\xi}{\xi+1})}\right)^{\frac{1}{2}}\int d^2z|z|^{\frac{2-2\xi}{\xi+1}}|1-z|^{\frac{2-2\xi}{\xi+1}}, \qquad (15)$$

$$C^{\Phi_{15}(1)}_{\Psi\Psi}(r) = -g \frac{\gamma(\frac{\xi}{\xi+1})\gamma(\frac{2}{\xi+1})}{\left(\gamma(\frac{3\xi}{\xi+1})\gamma(\frac{4\xi}{\xi+1})\gamma(\frac{2-2\xi}{\xi+1})\gamma(\frac{2-3\xi}{\xi+1})\right)^{\frac{1}{2}}} \times$$

$$\times \int d^2 z |z|^{\frac{2\xi}{\xi+1}} |1-z|^{\frac{2\xi}{\xi+1}}, \quad (16)$$

$$C^{\Phi(1)}_{\Psi\Psi}(r) = -g r^{\frac{\xi+4}{\xi+1}} \frac{\gamma(\frac{2}{\xi+1})}{\gamma^2(\frac{1}{\xi+1})} \lim_{\varepsilon\to\infty} \times$$

$$\times \int d^2x d^2y |x|^{-\frac{2\xi}{\xi+1}} |1-x|^{-\frac{2\xi}{\xi+1}} |y|^{\frac{2\xi}{\xi+1}-2\varepsilon} |1-y|^{\frac{2\xi}{\xi+1}-2\varepsilon} |x-y|^{-\frac{4\xi}{\xi+1}}. \quad (17)$$

In obtaining (17), we change the IR regularization via the system size R to the analytical regularization inserting the factor $|y|^{-2\varepsilon}|1-y|^{-2\varepsilon}$ in the integral and letting ϵ tend to 0 after afterward. The first two integrals are evaluated using the formula

$$\int d^2 z\ |z|^{2p} |z-1|^{2q} = \frac{\pi\gamma(p+1)\gamma(q+1)}{\gamma(p+q+2)}. \quad (18)$$

The third can be taken using the change of the variables $x \to \frac{x}{x-y}$ $y \to \frac{x-1}{x-y}$. We obtain the integral

$$\int d^2x d^2y |x|^{-\frac{2\xi}{\xi+1}} |y|^{-\frac{2\xi}{\xi+1}} |1-x|^{\frac{2\xi}{\xi+1}} |1-y|^{\frac{2\xi}{\xi+1}} |x-y|^{\frac{4\xi}{\xi+1}-6},$$

which can be evaluated using the well-known formula [5], [11],

$$\int d^2x d^2y |x|^{2a} |y|^{2a} |1-x|^{2b} |1-y|^{2b} |x-y|^{4c} =$$

$$= 2\pi^2 \frac{\gamma(2c)}{\gamma(c)} \frac{\gamma(1+a)\gamma(1+b)\gamma(1+a+c)\gamma(1+b+c)}{\gamma(2+a+b+c)\gamma(2+a+b+2c)} \quad (19)$$

Collecting all together, we obtain

$$\langle \Psi(r)\Psi(0)\rangle = r^{\frac{2-\xi}{\xi+1}} \left[A(r) - B(r)Q(1-2\xi) + D(r)Q(1-4\xi)\right]. \quad (20)$$

The r-dependent functions $A(r)$, $B(r)$, and $D(r)$ have the forms

$$A(r) = 1 - \frac{\xi^2}{4(\xi+3)^2} \frac{\gamma^2(\frac{\xi}{2})}{\gamma^2(\frac{1+\xi}{2})} (mr)^4$$

$$- \frac{(\xi+1)^2}{(1-\xi)(1-2\xi)} \left(\frac{\gamma^2(\frac{\xi}{\xi+1})\gamma^5(\frac{2}{\xi+1})\gamma(\frac{3\xi}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})\gamma(\frac{2-\xi}{\xi+1})\gamma^2(\frac{4}{\xi+1})} \right)^{\frac{1}{2}} (mr)^{\frac{4}{\xi+1}},$$

$$B(r)=\left\{\left(\frac{\gamma(\frac{\xi}{\xi+1})\gamma(\frac{2}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})\gamma(\frac{2-\xi}{\xi+1})}\right)^{\frac{1}{2}}\times\right.$$

$$\times\left[1+\frac{(\xi+1)^2}{(\xi-1)^2(3\xi+1)^2}\frac{\gamma(\frac{3\xi}{2})\gamma(\frac{1-3\xi}{2})}{\gamma(\frac{\xi}{2})\gamma(\frac{1-\xi}{2})}(mr)^4\right]$$

$$\left.+\frac{\xi^2(1-\xi)^3}{4(\xi+1)^2(1-2\xi)}\left(\frac{\gamma^8(\frac{1-\xi}{1+\xi})\gamma^9(\frac{\xi}{\xi+1})\gamma(\frac{3\xi}{\xi+1})}{\gamma^2(\frac{2-2\xi}{\xi+1})}\right)^{\frac{1}{2}}(mr)^{\frac{4}{\xi+1}}\right\}(mr)^{\frac{2(\xi-1)}{\xi+1}},$$

$$D(r)\quad=\quad-\frac{\xi^2(1-\xi)}{4(1-2\xi)(3\xi+1)^2}\left(\frac{\gamma^7(\frac{\xi}{\xi+1})\gamma^4(\frac{1-\xi}{\xi+1})}{\gamma(\frac{4\xi}{\xi+1})\gamma(\frac{2-2\xi}{\xi+1})\gamma(\frac{2-3\xi}{\xi+1})}\right)^{\frac{1}{2}}(mr)^{\frac{8\xi}{\xi+1}}.$$

4 Long-distance expansion

As previously mentioned, the combination of the spectral decomposition together with form factor bootstrap approach is very useful for studying the IR behavior of correlation functions. For example, for spin fields, the corresponding spectral decomposition is

$$\langle\Psi(x)\Psi(0)\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\{a_j\}}\int\frac{d\beta_1\cdots d\beta_n}{(2\pi)^n}e^{-r\sum_j m_{a_j}\cosh\beta_j}\times$$
$$\times F_{a_n\cdots a_1}(\beta_n,\ldots,\beta_1)F_{a_1\cdots a_n}(\beta_1,\ldots,\beta_n),\quad(21)$$

where we introduce the matrix elements of the local operators in the basis of the asymptotic states

$$F_{a_1\cdots a_n}(\beta_1,\ldots,\beta_n)=\langle 0|\Psi(0)|\beta_1,\ldots,\beta_n\rangle_{a_1\cdots a_n}.\quad(22)$$

Form factors (22) are defined in this approach as a set of functions satisfying Smirnov's axioms [2] (also see [10]). Here, we write only the final answer for the form factors of particle 1 of the primaries in the perturbed model [14],

$$
\begin{aligned}
&\langle 0|\Phi_{1k}|0\rangle = \langle\Phi_{1k}\rangle\,,\\
&\langle 0|\Phi_{1k}|\beta\rangle = i\;C\frac{\sin\Big((k-1)\pi\xi/2\Big)}{\sin(\pi\xi)}\;\langle\Phi_{1k}\rangle\,,\\
&\langle 0|\Phi_{1k}|\beta_2,\beta_1\rangle = i^2C^2\frac{\sin^2\Big((k-1)\pi\xi/2\Big)}{\sin^2(\pi\xi)}\;R(\beta_1-\beta_2)\;\langle\Phi_{1k}\rangle\,,
\end{aligned}
$$

and so on. Here, the function $R(\beta)$ determining the rapidity-dependent part of the two-particle form factor is given explicitly as

$$
R(\beta) = \exp\Big\{4\int_0^\infty \frac{dt}{t}\frac{\sinh t\sinh\xi t\sinh(\xi+1)t}{\sinh^2 2t}\cosh 2(1-\frac{i}{\pi}\beta)t\Big\}\,,
$$

and the constant C is

$$
C^2 = 8\cos^2(\frac{\pi\xi}{2})\sin(\frac{\pi\xi}{2})\exp\Big(-\int_0^{\pi\xi}\frac{dt}{\pi}\frac{t}{\sin t}\Big)\,.
$$

5 Conclusion

The results of numerical calculations for the model $\mathcal{M}_{2,7}$ perturbed by the energy operator are shown in Fig. 1. In the UV expansion we retain the first few terms (13),being restricted by knowledge of VEV of the first descendents. In the IR expansion the form factors with no more than three particles are taken into account.

The excellent matching of UV and IR expansions confirms the conjectures on the particle spectra, exact S-matrix, and VEV. This also gives a good approximation for the correlation functions at all scales. The program described above can be extended to find correlation functions of other operators in perturbed models (descendent fields). To do this, the VEV and form factors for these fields should be found. This problem is still unsolved.

Acknowledgments

This work was supported in part by the grants RFBR-04-02016027, LS-2044.20032 and by the program “Elementary Particles” of the Russian Academy

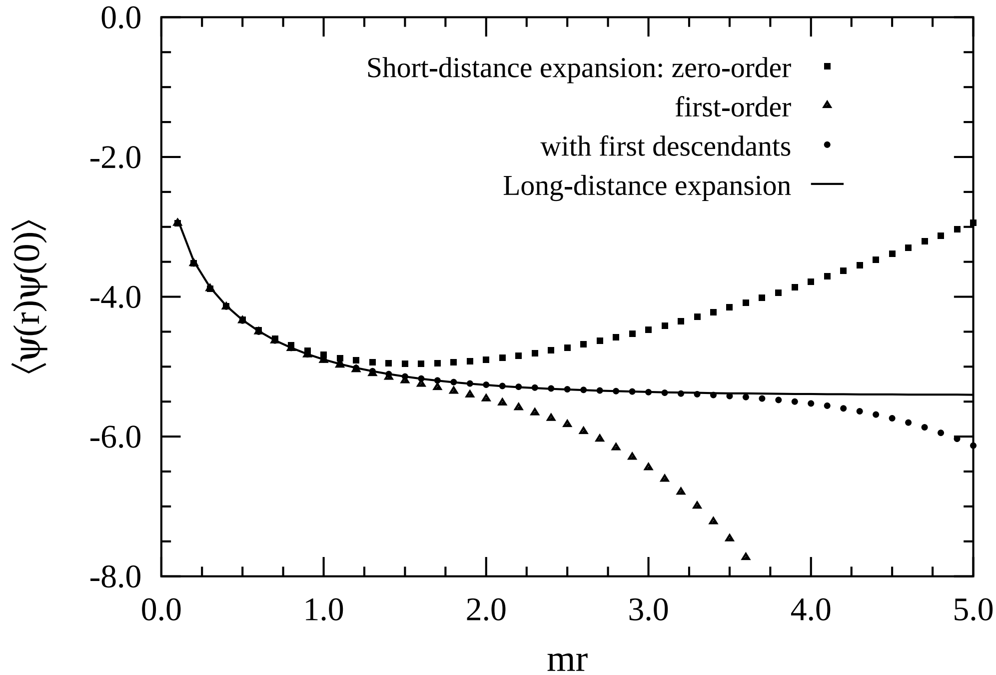

Figure 1: Correlation function of the two spin operators (in units $m^{\frac{8}{7}}$) in the minimal model $\mathcal{M}_{2,7}$ perturbed by the field Φ_{13}.

of Sciences. A. B. thanks to the organizers of the conference QUARKS 2004 and especially V. Rubakov for their kind hospitality. A. L. also thanks the Forshungcentrum Juelich for financial support in the framework of the Landau program and the Dynasty Foundation.

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