Helicity conservation in Born-Infeld theory

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Abstract

We prove that the helicity is preserved in the scattering of photons in the Born-Infeld theory (in 4d) on the tree level.

1. The Born-Infeld theory [1] introduced at the outset of the field theory epoch as a non-linear generalization of Maxwell theory has shown up recently as an effective theory of D-branes [2]. From the perturbative point of view, the Born-Infeld theory is quite complicated since its Lagrangian is an infinite power series in $F^{\mu\nu}$, so that there are infinitely many vertices, which makes the direct analysis in terms of Feynamn diagrams rather difficult. Nevertheless, it is possible to observe that the Born-Infeld theory in four dimensions possesses certain curious properties. Recall that the 4d Maxwell equations, $dF = d{*}F = 0$, are invariant with respect to the duality transformations: $\delta F = {*}F$. These transformations extend to the non-linear Born-Infeld theory [3].

Below, we make use of these transformations to deduce the conservation of helicity in the tree amplitudes of the Born-Infeld theory. We should emphasize that the duality transformations are defined on-shell only and do not correspond to a symmetry of the Born-Infeld Lagrangian. This is sufficient for the conservation law on the tree level, but there are no immediate consequences for the loop amplitudes (cf., however, a discussion in the conclusion). The remarkable helicity conservation (or, better to say, the selection rule) could indicate that the Born-Infeld theory is solvable in a sense.

2. The Born-Infeld theory is the theory of the abelian gauge potential A with the Lagrangian

$$L = \sqrt{det(g + F)}\,, \tag{1}$$

where $F = dA$ is the field strength 2-form and g is the (flat) space-time metric[1]. The corresponding field equations along with Bianchi identities read as

$$dF = 0\,, \quad d{*}G = 0\,, \tag{2}$$

where $G = \partial L/\partial F$. Let us consider the following infinitesimal transformation:

$$\delta F = {*}G\,. \tag{3}$$

Note that G is by definition a certain function of F, namely, $G(F) = \partial L/\partial F$, and therefore, the transformation law of G follows from that of F. In fact, one can verify [3] that

$$\delta G = {*}F\,. \tag{4}$$

It is now obvious that the field equations (2) are invariant with respect to such duality rotations. Note that the transformation $\delta F = {*}G$ is a non-local non-linear transformation of the fundamental field A of the theory and is defined only "on the mass shell": since $F = dA$ is equivalent to the identity $dF = 0$, the transformation law $\delta F = {*}G$ is well-defined as a transformation of the field A if only $d{*}G = 0$, i.e. on-shell.

3. In scattering theory we have to determine the quadratic (free-field) part of the Lagrangian and of the field equations. In our case, the latter are, of course, just the Maxwell equations for the gauge potential a,

$$df = 0\,, \quad d{*}f = 0\,, \tag{5}$$

where $f = da$. The plane wave solutions to these linear equations,

$$a = h\, e^{ik\cdot x}\,, \tag{6}$$

are defined by a 4-momentum k and a polarization 4-vector h, where $k^2 = 0$, $k \cdot x = k_\mu x^\mu$, and $k \cdot h = 0$ (the Lorentz gauge). It is convenient to use the basis of self-dual and anti-self-dual plane waves, corresponding to $f = \pm * f$ and assign a quantum number, the *self-duality number*, $s = 1$ to a plane wave (6) with $f = {*}f$ and $s = -1$ to a plane wave (6) with $f = - * f$. This basis of free-field states will be used to describe the scattering — as in- and out-states in a scattering process. We shall also adopt the convention that

[1] We shall consider this field theory in flat space-time only and adopt the convention $*^2 = 1$ for the duality operator $*$, disregarding the conventions related to the signature of the metric and of the reality of the fields: for our aims we need complex fields anyway.

the in- and out-states are distinguished by the sign of the time component of their 4-momenta k_μ, namely: $k_0 > 0$ corresponds to an in-coming particle, while $k_0 < 0$ to an out-going one. Note that the self-duality number s of a plane wave is nothing but the helicity: $s = 1$ means positive helicity for an in-coming photon and negative helicity for an out-going one (and opposite with $s = -1$).

4. We discuss now the scattering amplitudes. Let $\mathcal{A}_{tree}(a_1, a_2, a_3, \ldots)$ denote the generating function for the connected tree-level scattering amplitudes of an arbitrary number of particles. By definition, $\mathcal{A}_{tree}(a_1, a_2, a_3, \ldots)$ is a function of the plane waves $a_n = h_n\, e^{ik_n\cdot x}$, $n = 1, 2, 3, \ldots$, of definite self-dualities s_n , and is a formal sum of all connected tree amplitudes involving the in- and out-states described by the plane waves a_n . We are going to show that

the tree part of the amplitude vanishes unless the sum of self-dualities, $\sum_n s_n$, of all the scattering states is zero.

This means helicity conservation (in the tree scattering), for $\sum s_n = 0$ implies that the sum of helicities in the initial state is the same as in the final state.

Since each (connected, tree) amplitude corresponds to a term in the sum $\mathcal{A}_{tree}$ which is homogeneous with respect to a_n's, the above statement amounts to the invariance with respect to the phase rotations of the plane waves:

$$\mathcal{A}_{tree}(e^{is_1\alpha}a_1, e^{is_2\alpha}a_2, e^{is_3\alpha}a_3, \ldots) = \mathcal{A}_{tree}(a_1, a_2, a_3, \ldots) \tag{7}$$

It is this latter form of the conservation law which will be proved below.

5. In order to prove this we recall first the Lehmann-Symanzik-Zimmermann reduction formula[2] in the following special form:

$$\mathcal{A}_{tree}(a_1, a_2, a_3, \ldots) = \int d^4x\, a_1^\mu(x)\, \Box A_\mu^{ptb}(x|a_2, a_3, \ldots)\,, \tag{8}$$

where $a_1 = h_1\, e^{ik_1\cdot x}$ is the plane wave of a chosen, say, the first scattering particle; while A_μ^{ptb} is a certain solution to the classical (non-linear) field equations. This classical solution, defined in general elsewhere [4] under the name *perturbiner* is in fact a generating functional for the connected tree-level form-factors of the quantum field $A_\mu(x)$ (that is a formal sum of all such form-factors). For the present needs it is sufficient to mention that A_μ^{ptb} is a

[2]It corresponds to applying amputation on diagrams with one external leg off shell.

function of the space-time point x and the quantum numbers h_n, k_n of the scattering states, $A^{ptb} = A^{ptb}(x|\ldots, a_n, \ldots)$, and it is uniquely defined (up to a gauge transformation) as an expansion in powers of a_n's of the form

$$A^{ptb}(x) = \sum_n a_n(x) + \text{higher order terms in } {a_n}'\text{s}\,, \tag{9}$$

which obeys the classical field equation (2).

As a classical solution, A^{ptb} is subject to the duality transformations; for $F^{ptb} = dA^{ptb}$ and $G^{ptb} = G(F^{ptb})$, we have infinitesimally:

$$\delta F^{ptb} = *G^{ptb}\,, \quad \delta G^{ptb} = *F^{ptb}\,, \tag{10}$$

or, integrating to a finite rotation[3],

$$\begin{array}{rcl} (F^{ptb} + *G^{ptb}) & \to & e^{i\alpha}(F^{ptb} + *G^{ptb})\,, \\ (F^{ptb} - *G^{ptb}) & \to & e^{-i\alpha}(F^{ptb} - *G^{ptb})\,. \end{array} \tag{11}$$

On the other hand, the field $F^{ptb}(x)$ is uniquely determined by the quantum numbers of the waves $a_2, a_3, \ldots$. Therefore, the transformation (11) should correspond to a transformation of the plane-wave solutions. The latter can be found by observing the transformation of the first order term in the expansion (9) and turns out to be, of course,

$$a_n \to e^{is_n\alpha} a_n\,, \tag{12}$$

which corresponds to the duality rotation of (anti-)self-dual fields in Maxwell theory.

To summarise, we have the possibility to consider two types of transformations: the phase rotation of the plane waves, which is applicable to the functions of $a_1, a_2, a_3, \ldots$ (cf. eqs.(12),(7)) and the duality transformation (cf. eqs.(3),(11)), which is applicable to the solutions of the non-linear Born-Infeld equations. If we denote an infinitesimal phase rotation of a_n's by $\hat{\delta}$, we have to prove that

$$\hat{\delta}\mathcal{A}_{tree}(a_1, a_2, a_3, \ldots) = 0\,, \tag{13}$$

while for the perturbiner field A^{ptb} or, rather, for its field strength, F^{ptb}, we have that

$$\hat{\delta}F^{ptb} = \delta F^{ptb}\,, \tag{14}$$

[3]Note that we assume that $*^2 = 1$ and that all the fields are complex, so that $i = \sqrt{-1}$ in the exponent does not really matter much for the present needs.

where $\delta F^{ptb} = *G^{ptb}$ as earlier. Thus, to prove our proposition we may apply $\hat{\delta}$ to both sides of the reduction formula (8) and, then, use the equality (14) in the right hand side. Before doing this, we rewrite eq.(8) in a more convenient form:

$$\mathcal{A}_{tree}(a_1, a_2, a_3, \ldots) = \int *f_1 \wedge (F^{ptb} - G^{ptb})\,. \tag{15}$$

This latter form is obtained from eq.(8), in principle, by a formal integration by parts; one has only to be careful with poles corresponding to $k_1 + k_2 + k_3 + \ldots = 0$ [4]. The rest is easy now:

$$\begin{aligned} \hat{\delta}\mathcal{A}_{tree} &= \int \hat{\delta} * f_1 \wedge (F^{ptb} - G^{ptb}) + \int *f_1 \wedge (\delta F^{ptb} - \delta G^{ptb}) = \\ &= \int f_1 \wedge (F^{ptb} - G^{ptb}) + \int *f_1 \wedge (*G^{ptb} - *F^{ptb}) = \\ &= \int f_1 \wedge (F^{ptb} - G^{ptb}) + \int f_1 \wedge (G^{ptb} - F^{ptb}) = 0\,. \end{aligned} \tag{16}$$

9. We have just proved the helicity conservation in the Born-Infeld theory at the tree level. Note that this means vanishing of sums of certain diagrams, rather than vanishing of individual diagrams (helicity is not preserved by the vertices!). The latter statement applies only to the Lagrangian formulation corresponding to eq.(1) and may change when we switch to the Hamiltonian formulation. We are indebted to A. Vainshtein for this remark. At the one-loop level, it may be interesting to note that the unitarity implies vanishing of the imaginary parts of helicity violating amplitudes. The latter are then some *rational* functions of momenta and we would conjecture that it is possible to find them explicitly, analogously to what was found in the case of maximally helicity violating amplitudes in Yang-Mills theory ([5]).

For the future work we postpone also the question whether helicity conservation survives quantum corrections in maximally supersymmetric Born-Infeld theory, as well as applications of our observations to the string theory.

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[4] We have been careful!

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