Higher Symmetries of Toda Equations

Khazret S. Nirov[a*] and Alexander V. Razumov[b†]

[a]*Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect 7a, 117312 Moscow, Russia*

[b]*Institute for High Energy Physics, 142280 Protvino, Moscow Region, Russia*

Abstract

The symmetries of the simplest non-abelian Toda equations are discussed. The set of characteristic integrals whose Hamiltonian counterparts form a W-algebra, is presented.

1 Introduction

The main way to investigate nonlinear differential equations is through revealing and analysing their symmetries. The ideological foundation for this approach was laid at the end of the nineteenth and the beginning of twentieth centuries by S. Lie and E. Noether. Since then the most interesting and important results in this field of research have been obtained in development of the theory of integrable systems; for a quite exhaustive list of literature see monographs [1, 2].

A principal place among nonlinear integrable equations occupy two-dimensional systems which are representable as the zero-curvature condition for some connection on a trivial principal fiber bundle [2, 3, 4]. This transparent background allows for the most effective application of group-algebraic and differential-geometry methods to the study of such systems. A class of that sort, called the non-abelian Toda equations being a subclass of the Toda-type integrable systems [4], is considered in the present talk. Actually here we deal with the simplest examples of non-abelian Toda systems and concern only the symmetry aspects of the theory.

*E-mail: nirov@ms2.inr.ac.ru

†E-mail: razumov@mx.ihep.su

2 Non-abelian Toda systems

Let G be a real or complex matrix Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a $\mathbb{Z}$-gradation,

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m, \qquad [\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}.$$

Assume that the $\mathbb{Z}$-gradations under consideration is generated by a grading operator. It means that there exists an element $q \in \mathfrak{g}$, called the grading operator, such that

$$\mathfrak{g}_m = \{x \in \mathfrak{g} \mid [q, x] = mx\}.$$

It is known that any $\mathbb{Z}$-gradation of a semisimple Lie algebra is generated by a grading operator.

From the definition of a $\mathbb{Z}$-gradation it follows that the subspace $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$. Denote by G_0 the connected Lie group corresponding to this subalgebra. Let l be a positive integer such that the grading subspaces $\mathfrak{g}_m$ are trivial for $-l < m < 0$ and $0 < m < l$. In accordance with the group-algebraic approach [3, 4], the Toda equations are defined as the matrix differential equations

$$\partial_+ \left(\gamma^{-1} \partial_- \gamma\right) = [c_-, \gamma^{-1} c_+ \gamma]. \tag{2.1}$$

Here γ is a mapping from the manifold $\mathbb{R}^2$ with the coordinates denoted by x^- and x^+, c_- and c_+ are some fixed mappings from $\mathbb{R}^2$ to $\mathfrak{g}_{-l}$ and $\mathfrak{g}_{+l}$, respectively, satisfying the conditions

$$\partial_+ c_- = 0, \qquad \partial_- c_+ = 0.$$

As usually, we denote the partial derivatives over x^- and x^+ by ∂_- and ∂_+. There exist the so-called higher grading [5, 6] and multi-dimensional [7, 8] generalisations of the Toda systems.

If the $\mathbb{Z}$-gradation under consideration is such that the resulting Lie subalgebra $\mathfrak{g}_0$ and therefore the corresponding Lie subgroup G_0 are abelian, then the corresponding Toda system is called abelian. In any other case one has a non-abelian Toda system. There is a complete classification of the Toda systems associated with classical complex Lie groups [9].

The Toda equation (2.1) can be derived from the action functional [6]

$$S[\gamma] = S_{\mathrm{WZNW}}[\gamma] + S_{\mathrm{T}}[\gamma],$$

where $S_{\mathrm{WZNW}}[\gamma]$ is the action functional for the Wess–Zumino–Novikov–Witten (WZNW) model on the Lie group G_0 [10, 11], and $S_{\mathrm{T}}[\gamma]$ is a no-derivatives term

$$S_{\mathrm{T}}[\gamma] = \kappa \int \mathrm{d}x^- \mathrm{d}x^+ \, \mathrm{Tr} \left(\gamma^{-1} c_+ \gamma c_-\right).$$

Here κ is a free parameter entering the action of the WZNW model, and Tr is the appropriately normalized trace.

As a concrete example we consider in this talk some class of non-abelian Toda systems associated with the real general linear Lie group $\mathrm{GL}_{2n}(\mathbb{R})$ defined as follows. Let the Lie algebra $\mathfrak{gl}_{2n}(\mathbb{R})$ is supplied with the $\mathbb{Z}$-gradation generated by the grading operator

$$q = \frac{1}{2}\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},$$

where I_n is the unit $n \times n$ matrix. In this case $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ and $l = 1$. Choose the mappings c_- and c_+ as

$$c_- = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}, \qquad c_+ = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}.$$

The Lie group G_0 turns out to be isomorphic to the direct product of two copies of the general linear Lie group $\mathrm{GL}_n(\mathbb{R})$, and the mapping γ has the form

$$\gamma = \begin{pmatrix} \mathit{\Gamma}^{(1)} & 0 \\ 0 & \mathit{\Gamma}^{(2)} \end{pmatrix},$$

where $\mathit{\Gamma}^{(1)}$ and $\mathit{\Gamma}^{(2)}$ are mappings from $\mathbb{R}^2$ to $\mathrm{GL}_n(\mathbb{R})$. Toda equation (2.1) decomposes into two matrix differential equations

$$\partial_+ \left(\mathit{\Gamma}^{(1)-1} \partial_- \mathit{\Gamma}^{(1)} \right) = -\mathit{\Gamma}^{(1)-1} \mathit{\Gamma}^{(2)}, \quad \partial_+ \left(\mathit{\Gamma}^{(2)-1} \partial_- \mathit{\Gamma}^{(2)} \right) = \mathit{\Gamma}^{(1)-1} \mathit{\Gamma}^{(2)}.$$

The general solution to these equations was constructed in paper [12].

The action functional for the system under consideration is

$$S[\mathit{\Gamma}^{(1)}, \mathit{\Gamma}^{(2)}] = S_{\mathrm{WZNW}}[\mathit{\Gamma}^{(1)}] + S_{\mathrm{WZNW}}[\mathit{\Gamma}^{(2)}] + S_{\mathrm{T}}[\mathit{\Gamma}^{(1)}, \mathit{\Gamma}^{(2)}],$$

where $S_{\mathrm{WZNW}}[\mathit{\Gamma}^{(1)}]$ and $S_{\mathrm{WZNW}}[\mathit{\Gamma}^{(2)}]$ are the actions for the WZNW model on the above mentioned two copies of the group $\mathrm{GL}_n(\mathbb{R})$, and the no-derivatives term is now of the form

$$S_{\mathrm{T}}[\mathit{\Gamma}^{(1)}, \mathit{\Gamma}^{(2)}] = \kappa \int \mathrm{d}x^- \mathrm{d}x^+ \, \mathrm{tr} \left(\mathit{\Gamma}^{(1)-1} \mathit{\Gamma}^{(2)} \right).$$

Here tr means the usual trace.

Indubitable advantage given by the formulation of the system in terms of the action functional is that it allows us to proceed to the Hamiltonian description, which in turn provides a way to consider the symmetries of the system as the transformations generated by conserved charges. The construction of the Hamiltonian formalism can be performed in the spirit of paper [13].

3 Simplest symmetries

The action $S_{\mathrm{WZNW}}[\gamma]$ is invariant with respect to the transformation

$$\gamma(x^-, x^+) \to \lambda_+(x^+)\, \gamma(x^-, x^+)\, \lambda_-^{-1}(x^-), \tag{3.1}$$

where λ_- and λ_+ are arbitrary G_0-valued functions. The total action $S[\gamma]$ of the Toda system inherits only a part of this symmetry. Indeed, it is clear that transformation (3.1) leads again to the action of the Toda system with the mappings c_- and c_+ changed as follows:

$$c_- \to \lambda_- c_- \lambda_-^{-1}, \qquad c_+ \to \lambda_+ c_+ \lambda_+^{-1}.$$

Hence, if the matrix-valued functions c_- and c_+ are such that

$$\lambda_- c_- \lambda_-^{-1} = c_-, \qquad \lambda_+ c_+ \lambda_+^{-1} = c_+,$$

then (3.1) is a symmetry transformation for the Toda system under consideration. It is worthwhile to call this invariance a WZNW-type symmetry.

For the concrete example of a non-abelian Toda system introduced in the previous section the WZNW-type symmetry is realised as

$$\mathit{\Gamma}^{(1)}(x^-, x^+) \to \mathit{\Lambda}_+(x^+)\mathit{\Gamma}^{(1)}(x^-, x^+)\mathit{\Lambda}_-^{-1}(x^-), \tag{3.2}$$
$$\mathit{\Gamma}^{(2)}(x^-, x^+) \to \mathit{\Lambda}_+(x^+)\mathit{\Gamma}^{(2)}(x^-, x^+)\mathit{\Lambda}_-^{-1}(x^-). \tag{3.3}$$

It is clear that $S_{\mathrm{T}}[\gamma]$ is invariant with respect to these transformations.

The action of the WZNW model is invariant also with respect to the conformal transformations

$$x^- \to f^-(x^-), \qquad x^+ \to f^+(x^+)$$

acting on the mapping γ in accordance with the rule

$$\gamma(x^-, x^+) \to \gamma(f^-(x^-), f^+(x^+)).$$

The action of a Toda system is not invariant with respect to these transformations. Nevertheless, in the case where c_- and c_+ are constant mappings, one can demonstrate that the corresponding Toda system is invariant with respect to the slightly modified action of the conformal group. To this end, one can first get convinced that if the mapping γ satisfies the equation (2.1), then the mapping $\gamma'(x^-, x^+) = \gamma(f^-(x^-), f^+(x^+))$ satisfies the equation

$$\partial_+ \left(\gamma'^{-1} \partial_- \gamma'\right) = \partial_- f^- \partial_+ f^+ [c_-, \gamma'^{-1} c_+ \gamma'].$$

One can compensate the factor $\partial_- f^- \partial_+ f^+$ at the right hand side of this equation using the transformation (3.1). To this end, one should put

$$\lambda_- = \exp(+q\, l^{-1} \ln \partial_- f^-), \qquad \lambda_+ = \exp(-q\, l^{-1} \ln \partial_+ f^+),$$

where q is the grading operator. It leads to the relations

$$\lambda_- c_- \lambda_-^{-1} = (\partial_- f^-)^{-1} c_-, \qquad \lambda_+ c_+ \lambda_+^{-1} = (\partial_+ f^+)^{-1} c_+,$$

allowing exactly for the desirable cancellation of the superfluous factor. As the result, the conformally transformed mapping

$$\gamma'' = \exp(-q\, l^{-1} \ln \partial_+ f^+)\, \gamma' \exp(-q\, l^{-1} \ln \partial_- f^-) \tag{3.4}$$

satisfies the initial Toda equation. We see that the space of solutions of the Toda equations is invariant under the appropriately defined action of the group of conformal transformations [9].

4 Characteristic integrals and W-symmetries

The WZNW-type symmetry and the conformal symmetry do not exhaust all symmetries of a Toda system. To find additional symmetry transformations we can use the following procedure [14].

First we find some set of conserved charges. In the case under consideration we have an infinite set of conserved charges provided by the so-called characteristic integrals. In the Hamiltonian formalism conserved charges generate symmetry transformations. So, we construct the Lagrangian formulation for our system and then proceed to the corresponding Hamiltonian formulation. After that we consider the symmetry transformations generated by the Hamiltonian counterparts of the conserved charges associated with the characteristic integrals, and finally obtain their Lagrangian version. This allows us, in particular, to obtain the WZNW-type symmetry transformations and the conformal transformations discussed above.

A characteristic integral of a Toda system is, by definition, either a differential polynomial W of the Toda fields satisfying the relation

$$\partial_+ W = 0, \tag{4.1}$$

or a differential polynomial $\overline{W}$ of the Toda fields which satisfy the relation

$$\partial_- \overline{W} = 0. \tag{4.2}$$

By a differential polynomial we mean a polynomial function of the fields and their derivatives. The existence of the characteristic integrals, under appropriate conditions, guarantees the integrability of the Toda equations [15].

Let us treat the manifold $\mathbb{R}^2$ as a flat Riemannian manifold with the coordinates x^- and x^+ being light-front coordinates. The usual flat coordinates $x^0 = t$ and $x^1 = x$ are related to the light-front coordinates by the relation

$$x^0 = x^- + x^+, \qquad x^1 = -x^- + x^+.$$

Using these coordinates, we write equality (4.1) as

$$\partial_t W + \partial_x W = 0,$$

where $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$. Hence, the function W is a density of a conserved charge. Moreover, multiplying W by a function which depends only on x^- we again obtain a characteristic integral. Therefore, a characteristic integral generates an infinite set of densities of conserved charges. Similarly, multiplying a characteristic integral satisfying relation (4.2) by functions depending only on x^+ we again obtain an infinite set of densities of conserved charges.

It is clear that any differential polynomial of characteristic integrals is also a characteristic integral. Moreover, the Poisson bracket of the Hamiltonian counterparts of any two characteristic integrals is again a characteristic integral. Therefore, a necessary step in investigation of characteristic integrals is to show that they form a closed set with respect to the Poisson bracket, or, in other words, that they form an object called a W-algebra. The elements of a W-algebra generate the so-called W-symmetries being a nonlinear extension of the conformal symmetry. The study of such extensions was initiated by A. B. Zamolodchikov [16], for a review see [17].

At least two methods can be used to find the characteristic integrals. One of them consists in constructing a generating system of pseudo-differential operators [18, 19, 20, 21], and the other one is the Drinfeld–Sokolov highest weight gauge [22, 20, 21]. In fact, both of them are based on the representation of the Toda equations as the zero curvature condition of some connection on the trivial principal fibre bundle $\mathbb{R}^2 \times G \to \mathbb{R}^2$.

For the system under consideration the characteristic integrals appear as the matrix-valued quantities

$$\begin{aligned} W_1 &= -\kappa \left[\Gamma^{(1)-1} \partial_- \Gamma^{(1)} + \Gamma^{(2)-1} \partial_- \Gamma^{(2)} \right], \\ W_2 &= -\frac{\kappa^2}{2} \partial_- \left[\Gamma^{(1)-1} \partial_- \Gamma^{(1)} - \Gamma^{(2)-1} \partial_- \Gamma^{(2)} \right] \\ &\qquad + \kappa^2 \Gamma^{(1)-1} \partial_- \Gamma^{(1)} \Gamma^{(2)-1} \partial_- \Gamma^{(2)}, \end{aligned}$$

satisfying relation (4.1), and the matrix-valued quantities

$$\begin{aligned}
\overline{W}_1 = & -\kappa \left[\partial_+ \mathit{\Gamma}^{(1)} \mathit{\Gamma}^{(1)-1} + \partial_+ \mathit{\Gamma}^{(2)} \mathit{\Gamma}^{(2)-1}\right], \\
\overline{W}_2 = & -\frac{\kappa^2}{2} \partial_+ \left[\partial_+ \mathit{\Gamma}^{(1)} \mathit{\Gamma}^{(1)-1} - \partial_+ \mathit{\Gamma}^{(2)} \mathit{\Gamma}^{(2)-1}\right] \\
& \qquad + \kappa^2 \partial_+ \mathit{\Gamma}^{(2)} \mathit{\Gamma}^{(2)-1} \partial_+ \mathit{\Gamma}^{(1)} \mathit{\Gamma}^{(1)-1},
\end{aligned}$$

satisfying relation (4.2).

Denote the Hamiltonian counterparts of the above described characteristic integrals by $\mathcal{W}_1$, $\mathcal{W}_2$ and $\overline{\mathcal{W}}_1$, $\overline{\mathcal{W}}_2$. It can be shown that they are the generators of a W-algebra with respect to the Poisson bracket [14].

To find the form of the infinitesimal symmetry transformations generated by the characteristic integrals we consider first the quantity

$$\mathcal{W}_\varepsilon(t) = \int \mathrm{d}x \, \mathrm{tr} \left[\varepsilon_1(t,x) \mathcal{W}_1(t,x) + \varepsilon_2(t,x) \mathcal{W}_2(t,x)\right],$$

where ε_1 and ε_2 are arbitrary infinitesimal matrix-valued functions on $\mathbb{R}^2$ satisfying the relations

$$\partial_+ \varepsilon_1 = \partial_t \varepsilon_1 + \partial_x \varepsilon_1 = 0, \qquad \partial_+ \varepsilon_2 = \partial_t \varepsilon_2 + \partial_x \varepsilon_2 = 0.$$

The infinitesimal transformations of an observable $F(t)$ generated by $\mathcal{W}_\varepsilon(t)$ are given by the relation

$$\delta F(t) = \{\mathcal{W}_\varepsilon(t), F(t)\}.$$

For the mappings $\mathit{\Gamma}^{(1)}$ and $\mathit{\Gamma}^{(2)}$ these transformations written in the Lagrangian form are [14]

$$\delta_\varepsilon \mathit{\Gamma}^{(1)} = \mathit{\Gamma}^{(1)} \varepsilon_1 - \kappa \, \mathit{\Gamma}^{(1)} \mathit{\Gamma}^{(2)-1} \partial_- \mathit{\Gamma}^{(2)} \varepsilon_2 - \frac{\kappa}{2} \mathit{\Gamma}^{(1)} \partial_- \varepsilon_2, \tag{4.3}$$

$$\delta_\varepsilon \mathit{\Gamma}^{(2)} = \mathit{\Gamma}^{(2)} \varepsilon_1 - \kappa \, \mathit{\Gamma}^{(2)} \varepsilon_2 \mathit{\Gamma}^{(1)-1} \partial_- \mathit{\Gamma}^{(1)} + \frac{\kappa}{2} \mathit{\Gamma}^{(2)} \partial_- \varepsilon_2. \tag{4.4}$$

Similarly, introducing the quantity

$$\overline{\mathcal{W}}_\varepsilon(t) = \int \mathrm{d}x \, \mathrm{tr} \left[\bar{\varepsilon}_1(t,x) \overline{\mathcal{W}}_1(t,x) + \bar{\varepsilon}_2(t,x) \overline{\mathcal{W}}_2(t,x)\right],$$

where the infinitesimal matrix-valued functions $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ satisfy the relations

$$\partial_- \bar{\varepsilon}_1 = \partial_t \bar{\varepsilon}_1 - \partial_x \bar{\varepsilon}_1 = 0, \qquad \partial_- \bar{\varepsilon}_2 = \partial_t \bar{\varepsilon}_2 - \partial_x \bar{\varepsilon}_2 = 0,$$

we come to the following expressions for the infinitesimal transformations [14]

$$\delta_{\bar{\varepsilon}} \mathit{\Gamma}^{(1)} = \bar{\varepsilon}_1 \, \mathit{\Gamma}^{(1)} - \kappa \, \bar{\varepsilon}_2 \, \partial_+ \mathit{\Gamma}^{(2)} \, \mathit{\Gamma}^{(2)-1} \, \mathit{\Gamma}^{(1)} - \frac{\kappa}{2} \, \partial_+ \bar{\varepsilon}_2 \, \mathit{\Gamma}^{(1)}, \tag{4.5}$$

$$\delta_{\bar{\varepsilon}} \mathit{\Gamma}^{(2)} = \bar{\varepsilon}_1 \, \mathit{\Gamma}^{(2)} - \kappa \, \partial_+ \mathit{\Gamma}^{(1)} \, \mathit{\Gamma}^{(1)-1} \, \bar{\varepsilon}_2 \, \mathit{\Gamma}^{(2)} + \frac{\kappa}{2} \, \partial_+ \bar{\varepsilon}_2 \, \mathit{\Gamma}^{(2)}. \tag{4.6}$$

One can verify that transformations (4.3), (4.4) and (4.5), (4.6) are really symmetry transformations for the Toda system under consideration. Putting $\varepsilon_2 = 0$ and $\bar{\varepsilon}_2 = 0$ we obtain infinitesimal version of the transformations described by relations (3.2), (3.3).

The conformal transformations (3.4) are generated by the Hamiltonian counterparts of the nonzero components of the conformally improved energy-momentum tensor which are connected with the characteristic integrals as

$$T'_{--} = \frac{1}{\kappa} \operatorname{tr} \left[W_1^2 - 2W_2 \right], \qquad T'_{++} = \frac{1}{\kappa} \operatorname{tr} \left[\overline{W}_1^2 - 2\overline{W}_2 \right].$$

Note that the characteristic integrals W_1, $\overline{W}_1$ and W_2, $\overline{W}_2$ have the conformal spins 1 and 2 respectively [14].

Along the same lines one can investigate also some non-abelian Toda systems related to the real symplectic Lie group $\mathrm{Sp}_n(\mathbb{R})$ [14]. Actually these systems can be considered as the reduction of the systems defined above to the case where

$$\mathit{\Gamma}^{(1)} = (\mathit{\Gamma}^{(2)T})^{-1} = \mathit{\Gamma},$$

where T denotes the transposition with respect to the main skew diagonal. The Toda equations take in this case the form

$$\partial_+ (\mathit{\Gamma}^{-1} \partial_- \mathit{\Gamma}) = -(\mathit{\Gamma}^T \mathit{\Gamma}^{-1}). \tag{4.7}$$

In the case $n = 1$ denoting $\mathit{\Gamma} = \exp F$ one comes to the equation

$$\partial_+ \partial_- F = -\exp(-2F)$$

that is the well known Liouville equation. Therefore, it is natural to call equation (4.7) non-abelian Liouville equation.

5 Some deliberation

It would be very interesting to extend our consideration to non-abelian Toda equations based on $\mathbb{Z}$-gradations different from ones we have considered here [9]. For this, one should first construct the corresponding characteristic integrals. Of special interest are higher grading Toda-type systems including matter fields. One of the serious barriers to be overcome in this way is that such systems, in general, cannot be treated with the help of local Lagrangians [6], and so, the construction of conventional Hamiltonian descriptions seems to be rather problematic.

It is worth to note that the generators of the W-algebras considered above have conformal spin 1 or 2 only. Nevertheless, we have nonlinear defining relations. As far as we know it is a new phenomenon in the theory of W-algebras.

Another important direction leads from our work to the problem of quantization of non-abelian Toda systems. There, one cannot avoid the questions on quantum counterparts of our W-algebras and on the very meaning of the W-symmetries at the quantum level.

The work of A.V.R. was supported in part by the Russian Foundation for Basic Research under grant #01–01–00201 and the INTAS under grant #00–00561.

References

[1] P. J. Olver, *Applications of the Lie Groups to Differential Equations* (Springer, New York, 1986).

[2] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, (Springer, New York, 1987).

[3] A. N. Leznov and M. V. Saveliev, *Group-theoretical Methods for Integration of Nonlinear Dynamical Systems* (Birkhauser, Basel, 1992).

[4] A. V. Razumov and M. V. Saveliev, *Lie Algebras, Geometry, and Toda-type Systems* (Cambridge University Press, Cambridge, 1997).

[5] A. N. Leznov, The internal symmetry group and methods of field theory for integrating exactly soluble dynamic systems, In: *Group Theoretical Methods in Physics*, Proc. of the 1982 Zvenigorod seminar (New York, Harwood, 1985) 443–457.

[6] J.–L. Gervais and M. V. Saveliev, Higher grading generalizations of the Toda systems, Nucl. Phys. **B453** (1995) 449–476; `arXiv:hep-th/9505047`.

[7] A. V. Razumov and M. V. Saveliev, On some class of multidimensional nonlinear integrable systems, In: *Second International Sakharov Conference in Physics*, eds. I. M. Dremin and A. M. Semikhatov (World Scientific, Singapore, 1997) 547–551; `arXiv:hep-th/9607017`.

[8] A. V. Razumov and M. V. Saveliev, Multi-dimensional Toda-type systems, Theor. Math. Phys. **112** (1997) 999–1022; `arXiv:hep-th/9609031`.

[9] A. V. Razumov, M. V. Saveliev and A. B. Zuevsky, Non-abelian Toda equations associated with classical Lie groups, In: *Symmetries and Integrable Systems*, Proc. of the Seminar, ed. A. N. Sissakian (JINR, Dubna, 1999) 190–203; `arXiv:math-ph/9909008`.

[10] S. P. Novikov, The hamiltonian formalism and a multi-valued analogue of Morse theory, Russian Math. Surveys **37:5** (1982) 1–56.

[11] E. Witten, Nonabelian bosonization in two dimensions, Commun. Math. Phys. **92** (1984) 455–472.

[12] A. V. Razumov and M. V. Saveliev, Maximally non-abelian Toda systems, Nucl. Phys. **B494** (1997) 657–686; `arXiv:hep-th/9612081`.

[13] P. Bowcock, Canonical quantization of the gauged Wess–Zumino model, Nucl. Phys. **B316** (1989) 80–100.

[14] Kh. S. Nirov and A. V. Razumov, *to appear elsewhere*.

[15] A. N. Leznov, V. G. Smirnov and A. B. Shabat, Internal symmetry group and integrability conditions of two-dimensional dynamic systems, Theor. Math. Phys. **51** (1982) 322–333.

[16] A. B. Zamolodchikov, Infinite additional symmetries in two-dimensional conformal quantum field theory, Theor. Math. Phys. **65** (1985) 1205–1213.

[17] P. Bouwknegt and K. Schoutens, $\mathcal{W}$-symmetry in conformal field theory, Phys. Rep. **223** (1993) 183–276; `arXiv:hep-th/9210010`.

[18] A. Bilal and J.–L. Gervais, Extended $c = \infty$ conformal systems from classical Toda field theories, Nucl. Phys. **B314** (1989) 646–686.

[19] A. Bilal and J.–L. Gervais, Systematic construction of conformal theories with higher spin Virasoro symmetries, Nucl. Phys. B **B318** (1989) 579–642.

[20] J. Balog, L. Fehér, L. O'Raifeartaigh, P. Forgacs and A. Wipf, Toda theory and W-algebra from a gauged WZNW point of view, Ann. Phys. **203** (1990) 76–136.

[21] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, On Hamiltonian reductions of the Wess–Zumino–Novikov–Witten theories, Phys. Rep. **222** (1992) 1–64.

[22] V. G. Drinfeld and V. V. Sokolov, Lie algebras and equations of Korteweg–de Vries type, J. Sov. Math. **30** (1984) 1975–2036.