Ultraviolet Fixed Points in Gauge and SUSY Field Theories in Extra Dimensions

D.I.Kazakov

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia

and

Institute for Theoretical and Experimental Physics, Moscow, Russia

Abstract

We consider gauge field theories in $D > 4$ following the Wilson RG approach and show that they possess the ultraviolet fixed points where the gauge coupling is dimensionless in any space-time dimension. At the fixed point the anomalous dimensions of the field and vertex operators are known exactly. These fixed points are nonperturbative and correspond to conformal invariant theories. The same phenomenon also happens in supersymmetric theories with the Yukawa type interactions.

1 Introduction

Nowadays it became popular to consider theories in extra dimensions as possible candidates for models of physics beyond the Standard Model. (See e.g. Ref.[1, 2] and references therein.) One may wonder whether this extra dimensional theory can be considered as a consistent QFT in any sense. Since by general power counting it is nonrenormalizable, it looks hardly possible.

One way to consider an extra dimensional theory is the Kaluza-Klein approach. In this case, one takes the Fourier transform over the extra dimensions and obtains an infinite tower of states with quantized masses. Then one has to sum over all the states. This sum is usually divergent and a special prescription is needed to regularize it. Following this approach the divergences in D=5 SUSY theory have been studied in [3, 4, 5] for the scalar effective potential. Some cancellations of UV divergences have been found. Doubtfully, however, that this approach solves the problem of nonrenormalizability in extra dimensions.

In principle, there is a chance that all the UV divergences cancel each other, like it takes place in N=4, 2 and even N=1 SUSY field theories in D=4 [6], and one has a consistent theory. This possibility has been studied in the literature [7, 8, 9, 10]. Though at lower orders the divergences indeed cancel on shell [7, 8, 10], in higher orders they may well appear being unprotected by any symmetry [9].

In what follows we first remind the situation with the UV divergences in SUSY gauge theories in extra dimensions in the lowest order and then discuss an alternative

approach based on the Wilson renormalization group fixed points. The latter one is applied to the usual as well as supersymmetric gauge theories and exploits the nonperturbative RG fixed points for $D > 4$.

2 One-loop UV divergences in SUSY theories for arbitrary D.

Consider the one-loop vacuum polarization diagram in a non-Abelian gauge theory. It can be evaluated in arbitrary dimension using the technique of dimensional regularization. The result in the background field formalism is (we omit the transverse polarization tensor)

$$\begin{aligned}\Pi(p^2) &= (-)^{[D/2]}\frac{\Gamma(2-D/2)\Gamma^2(D/2)}{\Gamma(D)}\times \\ \{ &- \left[2\frac{8(D-1)-D'}{D-2}+\frac{(D-4)(D-1)\alpha(8-\alpha)}{2(D-2)}+\frac{4}{D-2}\right]C_2(G) \\ &+ 2^{[D'/2]}T(R)+\frac{4}{D-2}T(R)\Big\}\frac{1}{(p^2)^{2-D/2}}, \end{aligned} \tag{1}$$

where D is the dimension of integration and D' is the dimension of the fields corresponding to the Lorentz algebra. We present the result in an arbitrary α-gauge ($\alpha = 0$ corresponds to the Feynman gauge). The square bracket contains the gauge and ghost field contribution, and then follows those of spinor and scalar fields.

Taking $D = 4 - 2\varepsilon$ in eq.(1) one can reproduce the result for the logarithmic, quartic and sextic divergences in $D = 4, 6$ and 10, respectively. The singular part is proportional to

$$-(26-D')C_2(G)+2^{[D'/2]}T(R)+2T(R). \tag{2}$$

This is a gauge invariant expression of invariant operator $F^2_{\mu\nu}$.

Consider eq.(2) in particular cases corresponding to SUSY gauge theories in various dimensions taking the proper sets of the matter fields. The results are summarized below

$$\begin{array}{llll} D'=4 & N=1 & -22C_A+4C_A+4T_R+2T_R & =-6(3C_A-T_R), \\ & N=2 & -22C_A+4C_A+6C_A+12T_R & =-12(C_A-T_R), \\ & N=4 & -12C_A+12C_A & =0, \\ & & & \\ D'=6 & N=1 & -20C_A+8C_A+8T_R+4T_R & =-12(C_A-T_R), \\ & N=2 & -12C_A+12C_A & =0, \\ & & & \\ D'=10 & N=1 & -16C_A+16C_A & =0. \end{array}$$

One can see that when the matter field representations are chosen in a proper way, the leading divergences indeed cancel each other. Note that the $N = 1$ $D = 10$ case coincides with the $N = 2$ $D = 6$ and $N = 4$ $D = 4$ ones and the $N = 1$ $D = 6$ case coincides with the $N = 2$ $D = 4$ one as expected.

Return to logarithmic divergences in higher dimensions. Take $D = 6$ for definiteness. Due to the background field gauge invariance the divergent structures in the one-loop order can take one of the following forms:

$$\begin{aligned} I_1 &= TrD_\rho F_{\mu\nu} D_\rho F_{\mu\nu} \ , \\ I_2 &= TrD_\mu F_{\mu\nu} D_\rho F_{\rho\nu} \ , \\ I_3 &= TrD_\rho F_{\mu\nu} D_\mu F_{\rho\nu} \ , \\ I_4 &= TrF_{\mu\nu} F_{\nu\rho} F_{\rho\mu} \ . \end{aligned} \tag{3}$$

However, these invariants are not independent. Due to the relation $[D_\mu, D_\nu] = F_{\mu\nu}$ and the Bianchy identity $D_\mu F_{\nu\rho} + D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} = 0$, one has only 2 independent structures and can choose any of them. We take the first two. Then calculating the diagrams and extracting the contribution to two independent Lorentz structures one can find the coefficients of them. The result is

$$\frac{T_R - C_A}{3} \, TrD_\mu F_{\mu\nu} D_\rho F_{\rho\nu}. \tag{4}$$

One finds that the result for ALL the structures is proportional to $\sum T(R) - C_2(G)$, like for the qudratic divergences, and vanishes *off-shell*. Due to the fact that all the structures vanish we claim that all the one loop divergences in the gauge sector cancel for $\sum T(R) = C_2(G)$!

However, unlike the quadratic divergences, this result is gauge-dependent. In an arbitrary α-gauge eq.(4) looks like

$$\frac{T_R - C_A(1 + \alpha - \alpha^2/8)}{3} \, TrD_\mu F_{\mu\nu} D_\rho F_{\rho\nu}. \tag{5}$$

and the cancellation is not obvious anymore.

To get a gauge invariant statement, one has to go on-shell, i.e. to use the equations of motion. For the pure gauge case they are

$$D_\mu F_{\mu\nu} = \bar{\lambda}\gamma^\nu \lambda, \qquad \hat{D}\lambda = 0, \tag{6}$$

where λ is the gaugino field. Collecting the terms of effective action which transform into one another due to the equations of motion one has

$$...(D_\mu F_{\mu\nu})^2 + ...\bar{\lambda}\gamma^\nu D_\mu F_{\mu\nu}\lambda + ...(\bar{\lambda}\gamma^\nu\lambda)^2 = 0 \quad ! \ ,$$

that is one finds cancellation of the logarithmic divergences *on-shell* in any gauge.

In higher loops the following statements are valid:

1. The on-shell finiteness of the $D = 6$ $N = 1$ SUSY gauge theory is true in two loops as well. This has been checked by explicit calculation in components [7, 8];

2. Within the (constrained) superfield formalism it is possible to show that the allowed invariants vanish on-shell up to 2 loops. However, in higher loops the nonvanishing invariants exist [9]. The coefficients are not calculated but there is no known symmetry that might protect them.

Thus, our main conclusion is not optimistic: there is no big chance for the cancellation of logarithmic divergences for $D > 4$ even on-shell, i.e. the theory remains *perturbatively nonrenormalizable*.

3 Nonperturbative fixed point in gauge theories for D > 4.

We turn now to an alternative idea and look for nonperturbative possibilities to construct a viable higher dimensional theory. We follow the so-called Wilson Renormalization Group approach [11], but not in a scalar theory but in a gauge one. Our treatment of nonrenormalizable interactions follows that of M.Strassler [12, 13].

Consider first the usual gauge theory in D dimensions

$$\mathcal{L} = -\frac{1}{4} Tr F_{\mu\nu}^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]. \tag{7}$$

The fields and the coupling have the following canonical dimensions:

$$[A] = \frac{D-2}{2}\ , \quad [F] = \frac{D}{2}\ , \quad [g] = 2 - \frac{D}{2}.$$

This means that $D = 4$ is the critical dimension for the gauge interaction: the coupling here is dimensionless, the operators are marginal and the theory is renormalizable in a usual sense.

A dimensional analysis implies consideration of the dimensionless quantity

$$\tilde{g} \equiv g\mu^{D/2-2} \quad \Rightarrow \quad [\tilde{g}] = 0,$$

where μ is some scale.[1] Now one can go to the critical dimension $D = 4$ where the theory is renormalizable, and write down the RG equation for g

$$\mu \frac{d}{d\mu} g = g(\frac{1}{2}\gamma_A), \tag{8}$$

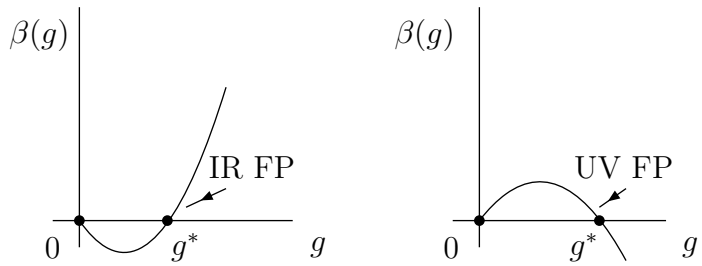
where γ_A is the gauge field anomalous dimension in the background field gauge. This gives, following Wilson's approach, the RG equation for $\tilde{g}$ which we consider in an arbitrary dimension D via the analytical continuation

$$\mu \frac{d}{d\mu} \tilde{g} = \tilde{g}\frac{1}{2}\gamma_A + \tilde{g}(\frac{D}{2} - 2) = \frac{\tilde{g}}{2}(\gamma_A + D - 4). \tag{9}$$

Eq.(9) has a fixed point. In fact, two of them

$$\begin{array}{ll} 1) & \tilde{g} = 0 \quad \rightarrow g = 0, \quad \gamma_A = 0, \\ 2) & g = g^*, \quad \gamma_A = 4 - D. \end{array}$$

The first one is trivial, this is the so-called Gaussian fixed point. It is perturbative. The second one is nonperturbative, it is the so-called Wilson-Fisher fixed point[11]. The anomalous dimension here is not small, it is integer. It is achieved at the value of the coupling which is unknown, though the value of the anomalous dimension is known *exactly*. Since the anomalous dimension in gauge theories, contrary to the scalar case, is negative, the fixed point of the second kind exists for $D > 4$. Remind that in scalar theories it exists for $D < 4$: one takes $D = 4 - \epsilon$, where $\epsilon \rightarrow 1$ or 2 and



Scalar Theory ($D < 4$) Gauge Theory ($D > 4$)

Figure 1:

performs the so-called $\epsilon-$expansion [11]. In the case of a scalar theory the FP is IR stable, while in a gauge theory it is UV stable (see Fig.1).

Consider the properties of the fixed point #2. Let us calculate the dimensions. One has for the field

$$[A] = \frac{D-2}{2} + \frac{1}{2}\gamma_A = \frac{D-2}{2} + \frac{4-D}{2} = 1$$

in any D. To calculate the dimension of the coupling, one has to consider the vertex $g\partial A[A, A]$ which gives

$$D = [g] + 1 + 3[A] + \gamma_V.$$

Since $\gamma_V = -\gamma_A$ in the background gauge, one obtains

$$[g^*] = D - 4 - \gamma_V = D - 4 + \gamma_A = 0 \quad \text{in any D} \ !$$

Thus, one has a dimensionless coupling at the fixed point that means renormalizability. *The theory at the fixed point is perturbatively nonrenormalizable, but nonperturbatively renormalizable!* (cf Ref.[13]). The existence of a renormalizable field theory beyond PT relies, in the sense of statistical physics, on the existence of a fixed point [15].

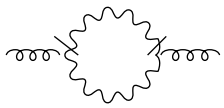
How can one understand this statement in terms of Feynman diagrams? Compare the two fixed points, the Gaussian one and the nonperturbative one

$g = 0$	$g = g^*$
$\widehat{AA} \sim \frac{1}{(x^2)^{\frac{D-2}{2}}}$	$\widehat{AA} \sim \frac{1}{(x^2)^1}$
$\int \frac{d^D x e^{ipx}}{(x^2)^{\frac{D-2}{2}}} \sim \frac{1}{p^2}$	$\int \frac{d^D x e^{ipx}}{(x^2)} \sim \frac{1}{(p^2)^{\frac{D-2}{2}}}$

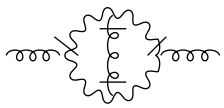
[1]Remind in dimensional regularization [14] $g_{Bare} = g\mu^\varepsilon$ in $D = 4 - 2\varepsilon$.

Thus, for instance, for $D = 6$ at the non-Gaussian fixed point the propagator behaves like $1/p^4$, i.e. much faster than in the usual case.

One can consider the diagrams with modified Feynman rules taking into account the anomalous dimensions. This corresponds to infinite summation of subgraphs. For the gauge propagator one has by power counting



$$\frac{D+2-2+2\gamma_V}{4-2\gamma_A} \;\Rightarrow\; D-4,$$



$$\frac{2D+4-2+4\gamma_V}{10-5\gamma_A} \;\Rightarrow\; 2D-8+\gamma_A = D-4,$$

$$\cdots\cdots\cdots \qquad\qquad \Rightarrow \quad D-4.$$

Hence, one has the same power in any loop, that is renormalizability. This is the consequence of dimensionless coupling at the fixed point.

One can try to construct an effective Lagrangian that describes these diagrams. In $D = 6$, as it is suggested by the one-loop calculation (4) and the behaviour of the propagator, it may be

$$\mathcal{L}_{eff} \sim \; Tr(D_\mu F_{\mu\nu})^2. \tag{10}$$

The effective Lagrangian (10) has some remarkable properties

- It has no scale, the coupling is dimensionless;
- It is scale (conformal) invariant;
- The anomalous dimensions of the field and vertices are known exactly[2];
- It is vanishing on-shell ($D_\mu F_{\mu\nu} = 0$).

At first sight, the effective Lagrangian (10) contains higher derivatives, and hence, ghosts. However, it is not clear for us how to define the spectrum of effective theory: is it the spectrum of the original Lagrangian or may be some new fields are adequate in this case?

4 Nonperturbative fixed point in SUSY theories for D > 4.

A similar phenomenon takes place in SUSY gauge theories. Again we start at the critical dimension $D = 4$ and use $N = 1$ superfields. Strictly speaking, they are $D = 4$ superfields; however, component notation is more cumbersome and what

[2]We do not know, however, any explicit procedure to calculate it except for the eq.(9).

we really need are the renormalizations in a critical dimension. So the superfield formalism here is not rigorous but useful.

The SUSY Lagrangian looks like (we omit the gauge fields for the moment)

$$\mathcal{L} = \int d^4\theta \; \bar{\Phi}_i \Phi_i + \int d^2\theta \; \mathcal{W} + h.c., \qquad \mathcal{W} = y\Phi_1\Phi_2\Phi_3. \tag{11}$$

Calculating the dimensions of the fields and the Yukawa coupling y, one has

$$[\mathcal{L}] = D, \quad [d\theta] = 1/2, \quad [\mathcal{W}] = D-1,$$
$$[\Phi] = \frac{D-2}{2}, \quad [y] = D - 1 - 3\frac{D-2}{2} = 2 - D/2.$$

Now we proceed as above. Introduce a dimensionless quantity $\tilde{y} = y\mu^{D/2-2}$ and write the RG equation for y in $D = 4$

$$\mu\frac{d}{d\mu}y = y(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_3), \tag{12}$$

where γ_i is the anomalous dimension of the matter field Φ_i. We use here the non-renormalization theorem in $D = 4$ which states that the anomalous dimension of the vertex is zero.

This allows us to get the RG equation for $\tilde{y}$

$$\mu\frac{d}{d\mu}\tilde{y} = \tilde{y}(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_3) + \tilde{y}(D/2-2) = \frac{\tilde{y}}{2}(\gamma_1 + \gamma_2 + \gamma_3 + D - 4). \tag{13}$$

This equation has two fixed points [12][3]

$$\begin{array}{ll} 1) & \tilde{y} = 0 \quad \to y = 0, \quad \gamma_i = 0, \\ 2) & y = y^*, \quad \gamma_i = (4-D)/3. \end{array}$$

One can see that for $D > 4$ the second fixed point requires the anomalous dimension to be negative. This is only possible in gauge theories. Hence, in fact one has to consider the gauge invariant SUSY theory where the nontrivial fixed point is (g^*, y^*). At this point the dimension of the Yukawa coupling is

$$[y^*] = D - 1 - 3\frac{D-2}{2} - \frac{\gamma_1 + \gamma_2 + \gamma_3}{2} = 0 \quad \text{in any D} \;\; !$$

Thus, again, we get a theory that is perturbatively nonrenormalizable, but nonperturbatively renormalizable at the nontrivial fixed point. At this point a theory possess all the properties mentioned above.

One may wonder whether this nontrivial fixed point is reachable. To see this, consider an $N = 1$ SUSY gauge theory and take the all-loop NSVZ β-function [17]. Extracting the γ_A one finds

$$\gamma_A = 2\alpha\frac{T_R - 3C_A - \frac{2}{r}\sum C_R\gamma_R}{1 - 2C_A\alpha},$$

[3]For a scalar SUSY theory in $D < 4$ this nonperturbative fixed point was earlier used in Ref.[16] to describe the self-avoiding random walk.

where $\alpha \equiv g^2/16\pi^2$.

For a pure SUSY Yang-Mills case one has the equation

$$\gamma_A = 2\alpha \frac{-3C_A}{1-2C_A\alpha} = 4 - D.$$

The solution is

$$\alpha^* = \frac{D-4}{D-1}\,\frac{1}{2C_A}.$$

This value is smaller than the pole value $\alpha_{pole} = 1/2C_A$. In particular, in $D = 6$ one has $\alpha^* = 1/5C_A$, as shown in Fig.2. Thus, the fixed point seems to be quite reachable.

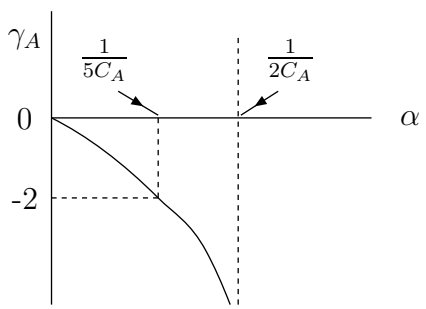


Figure 2:

5 Conclusion

Summarizing the analysis of the gauge and SUSY field theories in higher dimensions from the point of view of their renormalizability and consistency, we come to the following conclusions

- Perturbative finiteness in $D > 4$ seems not to be valid;
- Within the Wilson RG approach the nontrivial nonperturbative fixed points may lead to nonperturbative renormalizability;
- These theories may be related to PT renormalizable effective models which have to be found;
- At the fixed point the theory possesses the conformal invariance, and the anomalous dimensions are known exactly;
- These observed fixed points may be related to those in string theories in extra dimensions [18], and this may be the way to explore them.

Acknowledgements

I would like to thank M.Strassler, M.Shifman, V.Miransky, T.Jones, I.Jack, V.Rubakov, and A.Slavnov for useful discussions. I am grateful to the organizers of the conference "Quarks-2002" for their hospitality.

Financial support from RFBR grants # 02-02-16889 and # 00-15-96691 is kindly acknowledged.

References

[1] G.Altarelli and F. Feruglio, Phys.Lett. **B511** (2001) 257 (hep-ph/0102301).

[2] R.Barbieri, L.Hall and Y.Nomura, Phys.Rev. **D63** (2001) 105007 (hep-ph/0011311).

[3] I.Antoniadis, Phys.Lett. **B246** (1990) 377.

[4] H.-P.Nilles and D.Ghilencea, Phys.Lett. **B507** (2001) 327 (hep-ph/0103151); V.Di Clemente and Yu.A.Kubyshin, hep-th/0108117.

[5] I.Antoniadis, S.Dimopoulos and A.Pomarol, Nucl.Phys. **B544** (1999) 503; A.Delgado, A.Pomarol and M.Quiros, Phys.Rev. **D60** (1999) 095008; N.Arkani-Hamed, L.Hall, Y.Nomura, D.Smith and N. Weiner, Nucl.Phys. **B605** (2001) 81.

[6] D.R.T.Jones and L.Mezincescu, Phys.Lett. **B136** (1984) 242; S.Hamidi, J.Patera and J.H.Schwarz, Phys.Lett., **B141** (1984) 349; A.V.Ermushev, D.I.Kazakov and O.V.Tarasov, Nucl.Phys. **B281** (1987) 72; D.R.T.Jones, Nucl.Phys. **B279** (1986) 79; O.Piguet and K.Sibold, Phys.Lett. **B153** (1986) 373.

[7] N.Marcus and A.Sagnotti, Phys.Lett. **B135** (1984) 85; Nucl.Phys. **B256** (1985) 77.

[8] E.Fradkin and A.Tseytlin, Nucl.Phys. **B227** (1983) 252.

[9] P.S.Howe and K.Stelle, Phys.Lett. **B137** (1984) 175.

[10] D.Kazakov, hep-ph/0202150.

[11] K.G.Wilson, Phys.Rev. **B4** (1971) 3174, 3184; K.G.Wilson and J.B.Kogut, Phys.Rep. **12C** (1973) 75; K.G.Wilson ans M.E.Fisher, Phys.Rev.Lett. **28** (1972) 240.

[12] R.Leigh and M.Strassler, Nucl.Phys. **B447** (1995) 95; A.Nelson and M.Strassler, JHEP 0207 (2002) 021 (hep-ph/0104051).

[13] M.Strassler, Talk at the conf. "Continuous Advances in QCD-2002", Minnesota, May 2002.

[14] G.t'Hooft, Nucl.Phys.**B61** (1973) 455.

[15] J.Zinn-Justin, "Quantum Field Theory and Critical Phenomena", Clarendon Press, Oxford, 1989, p.549.

[16] D.Kazakov, Phys.Lett. **B215** (1988) 129.

[17] V.A.Novikov, M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Nucl. Phys. **B229** (1983) 381.

[18] N.Seiberg, Phys.Lett. **B388** (1996) 753; ibid **B390** (1997) 169.