String/Gauge Correspondence; View from the High Energy Side

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Abstract

We briefly review the recent progress concerning the application of the hidden integrability to the derivation of the stringy/brane picture for the high energy QCD.

1 Introduction

The explicit realization of the generic string/gauge correspondence program (see, for instance, [1]) remains the challenging problem during the last decades. It escaped the complete solution apart from the simplified two-dimensional example [2]. Some time ago itw system [19]. In the relevant

with the effective QCD Hamiltonian $\mathcal{H}_N$ acting on two-dimensional transverse coordinates of reggeons, $\vec{z}_k$ $(k = 1, ..., N)$ and their colour $SU(N_c)$ charges

$$\mathcal{H}_N = -\frac{\alpha_{\rm s}}{2\pi} \sum_{1\le i<j\le N} H_{ij}\ t_i^a t_j^a\,. \tag{3}$$

Here, the sum goes over all pairs of reggeons.

To get some insight into the properties of the $N-$reggeon states, it proves convenient to interpret the Feynman diagrams as describing a quantum-mechanical evolution of the system of N particles in the $t-$channel between two onia states $|A\rangle$ and $|B\rangle$

$$\sigma_{\rm tot}(s) = \sum_{N\ge 2} (\alpha_{\rm s} N_c)^N \langle A|\, {\rm e}^{\ln(1}$$

Here ϕ is the complex scalar field which generically develops the vacuum expectation value

$$\phi = \mathrm{diag}(\phi_1, ..., \phi_{N_c}) \,, \tag{9}$$

with $\mathrm{Tr}\,\phi = 0$. The gauge invariant order parameters $u_k = \langle 0|\mathrm{Tr}\,\phi^k|0\rangle$ parameterize the Coulomb branch of the vacuum manifold. They define a scale in the theory with respect to which one can discuss the issue of a low energy effective action. This action takes into account one-loop perturbative correction as well as the whole instanton series and it is governed by the Riemann surface of the genus $N_c - 1$ for $SU(N_c)$ gauge group. The same Riemann surface appears as the spectral curve of a classical integrable many-body system (see [24] for a review). The integrable system provides the natural explanation for the appearance of the meromorphic differential λ_{SW}, which turns out to be the action differential in the separated variables $\lambda_{\mathrm{SW}} = p\,dx$. Let us emphasize that for SUSY YM case the *classical* integrable system is relevant and the meaning of the quantum system and the corresponding spectrum for SUSY YM remains an open question. It should involve the quantization of the vacuum moduli space since the hamiltonian in the dynamical system is nothing but $H = u_2 = \langle \mathrm{Tr}\,\Phi^2\rangle$. Simultaneously, this parameter serves as the coordinate on the moduli space of the complex structures of the Riemann surfaces whiich means that the quantization of the integrable system is related with the quantization of the effective d=2 gravity.

Let us now consider a particular theory, namely the superconformal $\mathcal{N}$=2 SUSY YM with $N_f = 2N_c$ massless fundamental hypermultiplets [18, 25]. The corresponding integrable system is described by the spectral curve Σ_{N_c}

$$y^2 = P^2_{N_c}(x) - 4x^{2N_c}(1-\rho^2(\tau_{cl})) \,, \tag{10}$$

where $\rho^2(\tau_{cl})$ is some function of a coupling constant of the theory and the polynomial P_{N_c} depends on the coordinates on the moduli space $\vec{u} = (u_2, ..., u_{N_c})$

$$P_{N_c}(x) = \sum_{k=0}^{N_c} q_k(\vec{u})\, x^{N_c-k} = 2x^{N_c} + q_2\, x^{N_c-2} + ... + q_{N_c} \,, \tag{11}$$

where $q_0 = 2$, $q_1 = 0$ and other q_k are some known functions of $\vec{u}$. Their explicit form is not important for our purposes. The Seiberg-Witten meromorphic differential on the curve is given by

$$\lambda_{\mathrm{SW}} = p\,dx = \ln(\omega/x^{N_c})\,dx \,, \tag{12}$$

where $y = \omega - x^{2N_c}/\omega$.

From the point of view of integrable models the spectral curve corresponds to a classical XXX Heisenberg spin chain of length N_c with the spin zero at all sites (due to $q_1 = 0$) and parameter ρ related to the external magnetic field, or equivalently, to the twisted boundary conditions [20].

Comparing the spectral curve for the superconformal $\mathcal{N}$=2 SUSY YM with $N_f = 2N_c$ with the spectral curve for $N-$reggeon compound states in multi-colour QCD one observes that they coincide if we make the following identification

- The number of the reggeons $N = N_c$;
- The integrals of motion of multi-reggeon system are identified as the above mentioned functions $q_k(\vec{u})$ on the moduli space of the superconformal theory;
- The coupling constant of the gauge theory should be such that $\rho(\tau_{cl}) = 0$.

Under these three conditions the both theories fall into the same universality class. The third condition implies that the so-called strong coupling orbifold point on the moduli space corresponds to the Regge limit of multi-colour QCD. For instance, the strong coupling orbifold point $\rho(\tau_{\rm cl}) = 0$ describing the Odderon state in QCD occurs at

$$\tau_{\rm cl} = \frac{1}{2} + \frac{i}{2\sqrt{3}}\,. \tag{13}$$

Finally we would like to note that there is the following intriguing fact. In the case of $N_c = 2$ which corresponds on the QCD side to the BFKL Pomeron state, the effective coupling constant is given in the weak coupling regime by the expression

$$\tau_{\rm eff} = \tau_{\rm cl} + i\frac{4\ln 2}{\pi} + \sum_k c_k \cdot {\rm e}^{2i\pi k\tau_{\rm cl}}, \tag{14}$$

where the second term is due to a finite one-loop correction [27] and the rest is the sum of instanton contributions. It is amusing that this one-loop correction to the coupling constant

$$\frac{1}{g^2_{\rm eff}} = \frac{1}{g^2_{\rm cl}}\left[1 + \frac{g^2_{\rm cl}}{4\pi^2}4\ln 2\right] + ... \tag{15}$$

coincides with the expression for the intercept of the BFKL Pomeron after one identifies bare coupling constant in the superconformal theory with the t'Hooft coupling constant in QCD.

3 Brane picture for the Regge limit

Let us discuss now using the universality class arguments above a stringy/brane picture for the Regge limit in multi-colour QCD. To warm up we would like to recall the brane description of the low-energy dynamics of the $\mathcal{N} = 2$ SUSY YM. In the IIA framework the pure gauge theory is defined on the worldvolume of N_c D4 branes with the coordinates $(x_0, x_1, x_2, x_3, x_6)$ stretched between two NS5 branes with the coordinates $(x_0, x_1, x_2, x_3, x_4, x_5)$ and displaced along the coordinate x_6 by an amount inversely proportional to the coupling constant, $\delta x_6 = 1/g^2$ [29]. The coordinates at which the D4 branes intersect with the (x_4, x_5) complex plane define the vacuum expectation value of the scalar fields. This picture agrees perfectly with the RG behaviour of the coupling constant and yields the correct beta function in the gauge theory. The Riemann surface Σ discussed above describes the vacuum state of the theory and the spectrum of the stable BPS states. The lifting to the M theory picture leads to emergence of a single M5 brane with the worldvolume $R^4 \times \Sigma$ [28].

In our case, we also need to incorporate into this picture branes corresponding to the fundamental matter with $N_f = 2N_c$. There are two ways to do this: either using semi-infinite D4 branes lifted into M5 brane in the M theory, or using D6 branes which induce the nontrivial KK monopole background for the M2 brane wrapped on the Riemann surface [29]. As was shown in [20] in the latter case the resulting brane picture remains consistent with the integrable spin chain dynamics and we shall stick to this case.

The explicit metric of the KK background in the M theory involving (x_4, x_5, x_6, x_{10}) coordinates has the multi-Taub-NUT form

$$ds^2 = \frac{V}{4}d\vec{r}^2 + \frac{V^{-1}}{4}(d\tau + \vec{A}d\vec{r})^2 \tag{16}$$

where $\vec{r} = (x_4, x_5, x_6)$, $\tau = x_{10}$ and $\vec{A}$ is the Dirac monopole potential. The magnetic charge comes from the nontrivial twisting of S^1 bundle over R^3. The function V behaves as

$$V = 1 + \sum_{i=1}^{i=N_f} \frac{1}{|\vec{r} - \vec{r_i}|} \tag{17}$$

where $\vec{r}_i = (x_4^i, x_5^i, x_6^i)$ are the positions of six-branes. For a superconformal case one must have $x_4^i = x_5^i = 0$ and positions of sixbranes in x_6 direction are irrelevant.

Let us turn now to our proposal for the brane realization of the Regge limit. We shall explore the brane representations for the $N_f = 2N_c$ theory known in the IIA/M theory [29]. However unlike the SUSY case where the spectral curve is embedded in the internal "momentum" space the spectral curve of the noncompact spin chain is placed in the phase space involving both impact parameter plane as well as momenta. Consider the IIA/M type picture which is reminiscent to the realization of SYM theory via two NS5 and N_c D4 branes. We suggest that the coordinates involved in "IIA" picture are the transverse impact parameter coordinates x_1, x_2 and rapidity $\lambda = \ln k_+/k_-$. Transverse coordinates are analogue of (x_4, x_5) coordinates in SUSY case while rapidity substitutes the x_6 coordinate. Now let us make the next step and suggest that similar to SUSY case the single brane is wrapped around the spectral curve of XXX magnet and two "hadronic planes" together with N "Reggeonic strings" are just different projections of the single membrane with worldvolume $R \times \Sigma$. The coordinates involved into configurations are $x_1 + ix_2$ and $y = e^{-(\lambda + ix_{10})}$ where x_{10} is the "M-theory" coordinate.

Let us emphasize once again that the brane configuration for Regge limit contrary to SYM case partially involves the coordinate space. More precisely the geometry of $N_f = 2N_c$ theory is determined by the following parameter [29]

$$\xi = -\frac{4\lambda_+\lambda_-}{(\lambda_+ - \lambda_-)^2} \,. \tag{18}$$

Here λ_+ and λ_- are asymptotic positions of five-branes defined by the large x behaviour of the curve

$$\omega \propto \lambda_\pm x^{N_c} \,. \tag{19}$$

$\lambda_\pm$ can be found as roots of the equation

$$\lambda_\pm^2 + \lambda_\pm + \frac{1}{4}(1 - \rho^2) = 0 \tag{20}$$

Since Regge limit corresponds to the strong coupling orbifold point, $\rho = 0$, the value of ξ is fixed as $\xi = \infty$. This corresponds to the coinciding branes at infinity.

Finally, the M theory brane picture for the Regge limit involves M5 brane corresponding to the vacuum state of the QCD. We can not say how it is placed precisely as the minimal surface in the internal seven dimensional space since the corresponding geometry is unknown yet. The new ingredient - membrane share the time direction with the M5 brane and wrapping around the Riemann surface which is embedded into two-dimensional complex "phase space" with the multi-Taub-NUT metric determined by KK monopoles with the magnetic charge $2N$, which is a double number of reggeized gluons participating in the scattering process. The possible identification of the membrane above with the M2 brane deserves further investigation.

4 Quantum spectrum and S-duality

The $S-$duality is a powerful symmetry in the SUSY YM theory which allows us to connect the weak and strong coupling regimes. The effective coupling in this theory coincides with the modular parameter of the spectral curve of the underlying classical integrable model. As a consequence, the $S-$duality transformations in the gauge theory are translated into the modular transformations of the spectral curve describing complexified integrable system. A formulation of the $S-$duality in the latter system naturally leads to an introduction of the notion of the dual action [30].

So far the $S-$duality was well understood only for classical integrable models. In the case of multi-colour QCD in the Regge limit the situation is more complicated since the duality has to be formulated for a quantum integrable model. The integrals of motion take quantized set of values and the coordinates on the moduli space are not continuous anymore. Therefore the question to be answered is whether it is possible to formulate some duality transformations at the quantum level.

To study this question let us propose the WKB quantization conditions which are consistent with the duality properties of the complexified dynamical system whose solution to the classical equations of motion are described by the Riemann surface. We recall that the standard WKB quantization conditions involve the real slices of the spectral curve

$$\oint_{A_i} p\,dx = 2\pi\hbar(n_i + 1/2) \tag{21}$$

where n_i are integers and the cycles A_i correspond to classically allowed trajectories on the phase space of the system. In our case the coordinate x is complex and arbitrary point on the Riemann surface is classically allowed. As a result the general classical motion involves both $A-$ and $B-$cycles on the Riemann surface. This leads to the following generalized WKB quantization conditions for actions and dual actions

$$\mathrm{Re}\oint_{A_i} p\,dx = \pi\hbar n_i\,, \qquad \mathrm{Re}\oint_{B_i} p\,dx = \pi\hbar m_i\,, \tag{22}$$

where the "action" differential was defined in (12). Note that in the context of the SUSY YM this condition would correspond to the nontrivial constraints on the periods and on the mass spectrum of the BPS particles . It is clear that the WKB conditions (22) imply the duality $A_i \leftrightarrow B_i$ and $n_i \leftrightarrow m_i$.

Let us consider the quantization conditions (22) in the simple case of the Odderon $N = 3$ system. The spectral curve is a torus

$$y^2 = (2x^3 + q_2 x + q_3)^2 - 4x^6 \tag{23}$$

where q_2 is given by conformal spin while q_3 is the complex integral of motion to be quantized. The quantization conditions (22) read (for $\hbar = 1$)

$$\mathrm{Re}\, a(q_3) = \pi n, \qquad \mathrm{Re}\, a_D(q_3) = \pi m \tag{24}$$

where n and m are integer. These equations can be solved for large values of $q_3^2/q_2^3 \gg 1$, for which the expressions for the periods $a(q_3)$ and $a_D(q_3)$ are simplified considerably. The explicit evaluation of the integrals in this limit yields

$$a(q_3) = \frac{(2\pi)^2 q_3^{1/3}}{\Gamma^3(2/3)}\,, \qquad a_D(q_3) = a(q_3)\left(\frac{1}{2} + \frac{i}{2\sqrt{3}}\right) \tag{25}$$

Substituting these expressions into (24) one finds

$$q_3^{1/3} = \frac{\Gamma^3(2/3)}{2\pi}\left(\frac{\ell_1}{2} + i\frac{\sqrt{3}}{2}\ell_2\right)\,. \tag{26}$$

where $\ell_1 = n$ and $\ell_2 = n - 2m$. The WKB expressions (26) are in a good agreement with the exact expressions for quantized q_3 obtained from the numerical solutions of the Baxter equations in [26]. Note that WKB formulae can not be naively applied to the ground states discussed in [26, 31, 33].

Let us emphasize that the point at the moduli space corresponding to the degeneration of the torus for the Odderon case does not appear in the quantum spectrum. From the point of view of the gauge theory this means that the appearance of the massless states is forbidden.

In the general multi-reggeon case we have to consider the quantization conditions (22) on the Riemann surface of the genus $(N-2)$ which has the same number of the $A-$ and $B-$cycles. In result the spectrum of the integrals of motion q_3, ..., q_N is parametrized by two $(N-2)-$component vectors $\vec{n}$ and $\vec{m}$. In the SUSY YM case these vectors define the electric and magnetic charges of the BPS states.

In the Regge case the physical interpretation of $\vec{n}$ and $\vec{m}$ is much less evident. Let us first compare the electric quantum numbers in the two cases. In Regge case it corresponds to rotation in the coordinate space around the ends of the Reggeons. This picture fits perfectly with the interpretation of the electric charge in SUSY YM case. Indeed VEVs of the complex scalar take values on the complex plane which is the counterpart of the impact parameter plane and the rotation of the phase of the complex scalar is indeed the "electric rotation".

To get some guess concerning the "magnetic" quantum numbers it is instructive to check the geometrical picture behind them in the simplified "IIA" picture. All states corresponding to the "electric" degrees of freedom are related to fundamental strings encircling "reggeoinic" tubes and don't feel the hadronic planes. However the "magnetic" states as is well known from SUSY YM case are represented by the membrane stretched between two strings and two hadronic planes. Therefore these states are sensitive to hadronic quantum numbers. More detailed interpretation of "magnetic" degrees of freedom in the Regge regime has to be recovered.

5 Stringy/brane picture and the calculation of the anomalous dimensions

We have demonstrated that integrability properties of the Schrödinger equation for the compound state of Reggeized gluons give rise to the stringy/brane picture for the Regge limit in multi-colour QCD. There is another limit in which QCD exhibit remarkable properties of integrability. It has to do with the scaling dependence of the structure functions of deep inelastic scattering and hadronic light-cone wave functions in QCD. In the both cases, the problem can be studied using the Operator Product Expansion and it can be reformulated as a problem of calculating the anomalous dimensions of the composite operators of a definite twist. The operators of the lowest twist have the following general form

$$\begin{aligned} \mathcal{O}^{(2)}_{N,k}(0) &= (yD)^k \Phi_1(0)(yD)^{N-k}\Phi_2(0), \\ \mathcal{O}^{(3)}_{N,\boldsymbol{k}}(0) &= (yD)^{k_1} \Phi_1(0)(yD)^{k_2}\Phi_2(0)(yD)^{N-k_1-k_2}\Phi_3(0), \end{aligned} \tag{27}$$

where $\boldsymbol{k} \equiv (k_1, k_2)$ denotes the set of integer indices k_i, y_μ is a light-cone vector such that $y_\mu^2 = 0$. Φ_k denotes elementary fields in the underlying gauge theory and $D_\mu = \partial_\mu - iA_\mu$ is a covariant derivative. The operators of a definite twist mix under renormalization with each other. In order to find their scaling dependence one has to diagonalize the corresponding matrix of the anomalous dimension and construct linear combination of such operators, the so-called conformal operators

$$\mathcal{O}^{\rm conf}_{N,q}(0) = \sum_{\boldsymbol{k}} C_{\boldsymbol{k},\boldsymbol{q}} \cdot \mathcal{O}_{N,\boldsymbol{k}}(0)\,. \tag{28}$$

A unique feature of these operators is that they have an autonomous RG evolution

$$\Lambda^2 \frac{d}{d\Lambda^2}\, \mathcal{O}^{\rm conf}_{N,q}(0) = -\gamma_{N,q} \cdot \mathcal{O}^{\rm conf}_{N,q}(0)\,. \tag{29}$$

Here Λ^2 is a UV cut-off and $\gamma_{N,q}$ is the corresponding anomalous dimension depending on some set of quantum numbers $\boldsymbol{q}$ to be specified below. It turns out that the problem of calculating the spectrum of the anomalous dimensions $\gamma_{N,q}$ to one-loop accuracy becomes equivalent to solving the Schrödinger equation for the $SL(2,\mathbb{R})$ Heisenberg spin magnet. The number of sites in the magnet is equal to the number of fields entering into the operators under consideration.

To explain this correspondence it becomes convenient to introduce nonlocal light-cone operators

$$F(z_1, z_2) = \Phi_1(z_1 y)\Phi_2(z_2 y)\,, \qquad F(z_1, z_2, z_3) = \Phi_1(z_1 y)\Phi_2(z_2 y)\Phi_3(z_3 y)\,. \tag{30}$$

Here y_μ is a light-like vector ($y_\mu^2 = 0$) defining certain direction on the light-cone and the scalar variables z_i serve as a coordinates of the fields along this direction. The fields $\Phi_i(z_i y)$ are transformed under the gauge transformations and it is tacitly assumed that the gauge invariance of the nonlocal operators $F(z_i)$ is restored by including the Wilson lines between the fields in the appropriate (fundamental or adjoint) representation. The conformal operators appear in the OPE expansion of the nonlocal operators (30) for small $z_1 - z_3$ and $z_2 - z_3$.

The field operators entering the definition of $F(z_i)$ are located on the light-cone. This leads to the appearance of the additional light-cone singularities. They modify the renormalization properties of the nonlocal light-cone operators (30) and lead to nontrivial evolution equations which as we will show below become related to integrable chain models. We notice that there exists the following relation between the conformal three-particle operators (28) and the nonlocal operators (30)

$$\mathcal{O}^{\rm conf}_{N,q}(0) = \Psi_{N,q}(\partial_{z_1}, \partial_{z_2}, \partial_{z_3}) F(z_1, z_2, z_3)\bigg|_{z_i=0} , \tag{31}$$

where $\Psi_{N,q}(x_1, x_2, x_3)$ is a homogenous polynomial in x_i of degree N

$$\Psi_{N,q}(x_1, x_2, x_3) = \sum_{\boldsymbol{k}} C_{\boldsymbol{k},\boldsymbol{q}} \cdot x_1^{k_1} x_2^{k_2} x_3^{N-k_1-k_2} \tag{32}$$

with the expansion coefficients $C_{\boldsymbol{k},\boldsymbol{q}}$ defined in (28). The problem of defining the conformal operators is reduced to finding the polynomial coefficient functions $\Psi_{N,q}(x_i)$ and the corresponding anomalous dimensions $\gamma_{N,q}$. Using the renormalization properties of the nonlocal light-cone operators (30) one can show [14], that to the one-loop accuracy the QCD evolution equation for the conformal operators (30) can be rewritten in the form of a Schrödinger equation

$$\mathcal{H} \cdot \Psi_{N,q}(x_i) = \gamma_{N,q} \Psi_{N,q}(x_i) , \tag{33}$$

where the Hamiltonian $\mathcal{H}$ acts on the x_i-variables which are conjugated to the derivatives ∂_{z_i} and, therefore, have the meaning of light-cone projection $(y \cdot p_i)$ of the momenta p_i carried by particles described by fields $\Phi(z_i y)$.

For example when Φ_1 and Φ_2 are quark fields of the same chirality

$$F_{\alpha\beta}(z_1, z_2) = \sum_{i=1}^{N_c} (\bar{q}_i^{\uparrow} \not{y})_\alpha (z_1 y)(\not{y} q_i^{\uparrow})_\beta (z_2 y) \tag{34}$$

with $q_i^{\uparrow} = (1 + \gamma_5) q_i/2$, the two-particle Hamiltonian is given by

$$H_{12} = \frac{\alpha_{\rm s}}{\pi} C_F \left[H_{qq}(J_{12}) + 1/4 \right] , \qquad H_{qq}(J_{12}) = \psi(J_{12}) - \psi(2). \tag{35}$$

where $C_F = (N_c^2 - 1)/(2N_c)$. The eigenfunctions for this Hamiltonian are the highest weights of the discrete series representation of the $SL(2, R)$ group

$$\Psi_N^{(2)}(x_1, x_2) = (x_1 + x_2)^N C_N^{3/2} \left(\frac{x_1 - x_2}{x_1 + x_2} \right) \tag{36}$$

where $C_N^{3/2}$ are Gegenbauer polynomials. The corresponding eigenvalues define the anomalous dimensions of the twist-2 mesonic operators built from two quarks with the same helicity

$$\gamma_N^{(2)} = \frac{\alpha_{\rm s}}{\pi} C_F [\psi(N + 2) - \psi(2) + 1/4] = \frac{\alpha_{\rm s}}{\pi} C_F \left[\sum_{k=1}^{N} \frac{1}{k+1} + \frac{1}{4} \right] . \tag{37}$$

At large N this expression has well-known asymptotic behaviour $\gamma_N^{(2)} \sim \alpha_s C_F/\pi \ln N$.

It is conformal symmetry which dictates that the two-particle Hamiltonian is a function of the Casimir operator of the $SL(2,\mathbb{R})$ group, but it does not fix this function. The fact that this function turns out to be the Euler ψ-function leads to a hidden integrability of the evolution equations for anomalous dimensions of baryonic operators . Namely, for baryonic operator built from three quark fields of the same chirality

$$F_{\alpha\beta\gamma}(z_1,z_2,z_3) = \sum_{i,j,k=1}^{N_c} \epsilon_{ijk} (\not{y} q_i^{\uparrow})_\alpha (z_1 y)(\not{y} q_j^{\uparrow})_\beta (z_2 y)(\not{y} q_k^{\uparrow})_\gamma (z_3 y) \tag{38}$$

the evolution kernel is given by [13, 14]

$$\mathcal{H}^{(3)} = \frac{\alpha_s}{2\pi} \left\{ (1+1/N_c) \left[H_{qq}(J_{12}) + H_{qq}(J_{23}) + H_{qq}(J_{31}) \right] + 3C_F/2 \right\} \tag{39}$$

with H_{qq} given by (35). The Schrödinger equation (33) with the Hamiltonian defined in this way has a hidden integral of motion

$$q = i \left(\partial_{x_1} - \partial_{x_2}\right) \left(\partial_{x_2} - \partial_{x_3}\right) \left(\partial_{x_3} - \partial_{x_1}\right) x_1 x_2 x_3 \tag{40}$$

and, therefore, it is completely integrable. Similar to the Regge case, one can identify (39) as the Hamiltonian of a quantum XXX Heisenberg magnet of $SL(2,\mathbb{R})$ spin $j_q = 1$. The number of sites is equal to the number of quarks.

Based on this identification we shall argue now that the calculation of the anomalous dimensions can be formulated entirely in terms of Riemann surfaces which in turn leads to a stringy/brane picture. It is important to stress here the key difference between Regge and light-cone limits of QCD. In the first case the impact parameter space provides the complex plane for the Reggeon coordinates and we are dealing with a $(2+1)-$dimensional dynamical system. In the second case the QCD evolution occurs along the light-cone direction and is described by a $(1+1)-$dimensional dynamical system. As a consequence, in these two cases we have two different integrable magnets: the $SL(2,\mathbb{C})$ magnet for the Regge limit and the $SL(2,\mathbb{R})$ magnet for the light-cone limit. The evolution parameters ("time" in the dynamical models) are also different: the rapidity $\ln s$ for the Regge case and the RG scale $\ln \mu$ for the anomalous dimensions of the conformal operators.

Our approach to calculation of the anomalous dimensions via Riemann surfaces looks as follows. For concreteness, we shall concentrate on the evolution kernel (39). Similarly to the Regge case, one starts with the finite-gap solution to the classical equation of motion of the underlying $SL(2,\mathbb{R})$ spin chain and identifies the corresponding Riemann surface

$$\omega - \frac{x^6}{\omega} = 2x^3 - (N+2)(N+3)x + q\,, \qquad \omega = x^3\, \mathrm{e}^p \tag{41}$$

where q is the above mentioned integral of motion (40) and N is the total $SL(2,\mathbb{R})$ spin of the magnet, or equivalently the number of derivatives entering the definition of the conformal operator (28). Note that the Riemann surface corresponding to the three-quark operator has genus $g = 1$, while $g = 0$ for the twist 2 operators.

At the next step we quantize the Riemann surface in Sklyanin's approach [34]. We replace $p = i\partial/\partial x$ and impose the equation of the complex curve as the operator annihilating the Baxter function

$$\left(\mathrm{e}^{i\partial/\partial x} + \mathrm{e}^{-i\partial/\partial x}\right) x^3 Q(x) = \left[2x^3 - (N+2)(N+3)x + q\right] Q(x)\,. \tag{42}$$

This leads to the Baxter equation for the Heisenberg $SL(2,\mathbb{R})$ magnet of spin $j = 1$

$$(x+i)^3 Q(x+i) + (x-i)^3 Q(x-i) = \left[2x^3 - (N+2)(N+3)x + q\right] Q(x)\,. \tag{43}$$

Similar to the Baxter equation in the Regge case, this equation does not have a unique solution. To avoid this ambiguity one has to impose the additional conditions that $Q(x)$ should be polynomial in x. This requirement leads to the quantization of the integral of motion. The resulting polynomial solution $Q = Q_q(x)$ has the meaning of the one-particle wave function in the separated variables x which in the case of the $SL(2,\mathbb{R})$ magnet take arbitrary real values.

Given the polynomial solution to the Baxter equation (43), one can determine the eigenspectrum of the Hamiltonian (39) and in result the anomalous dimensions of the corresponding baryon operators

$$\gamma^{(3)}_{N,q} = \frac{\alpha_{\rm s}}{2\pi}\left[(1+1/N_c)\mathcal{E}_{N,q} + 3C_F/2\right] , \qquad \mathcal{E}_{N,q} = i\frac{Q'_q(i)}{Q_q(i)} - i\frac{Q'_q(-i)}{Q_q(-i)} \tag{44}$$

parameterized by the eigenvalues of the integral of motion (40) given by

$$q = -i\frac{Q_q(i) - Q_q(-i)}{Q_q(0)} \tag{45}$$

The solution to the Baxter equation simplifies greatly in the quasiclassical approximation which is controlled by the total $SL(2,\mathbb{R})$ spin of the system N. For $N \gg 1$ the spectrum of the integral of motion q is determined by the WKB quantization condition [23]

$$\oint_A p\, dx = 2\pi(n + 1/2) + \mathcal{O}(1/N) \tag{46}$$

where p was introduced in (42). Here integration goes over the A-cycle on the Riemann surface defined by the spectral curve (42). This cycle encircles the interval on the real $x-$axis on which $|\,\mathrm{e}^p\,| > 1$. Solving (46) one gets

$$q = \pm\frac{N^3}{\sqrt{27}}\left[1 - 3\left(n + \frac{1}{2}\right)N^{-1} + \mathcal{O}(N^{-2})\right] . \tag{47}$$

Taking into account that in this case $q_2 = -(N+3)(N+2)$ and $q_3 = q$ we conclude that for $N \to \infty$ the system is approaching the Argyres-Douglas point. Note also that the WKB quantization conditions (46) involve only the $A-$cycle on the Riemann surface and unlike the Regge case there is no $S-$duality in the quantum spectrum in the light-cone case.

Finally, the spectrum of the anomalous dimensions in the WKB approximation is given by [23, 14]

$$\mathcal{E}_{N,q} = 2\ln 2 - 6 + 6\gamma_E + 2\mathrm{Re}\sum_{k=1}^{3}\psi(1 + i\delta_k) + \mathcal{O}(N^{-6}), \tag{48}$$

where δ_k are defined as roots of the following cubic equation:

$$2\delta_k^3 - (N+2)(N+3)\delta_k + q = 0 \tag{49}$$

and q satisfies (47).

What can we learn about stringy picture from this information about anomalous dimensions? Let us remind that in the spirit of string/gauge fields correspondence the anomalous dimensions of gauge field theory operators coincide with excitation energies of a string in some particular background. An important lesson that we have learned from the analysis of the two- and three-quark operators is that in the first case the anomalous dimensions are uniquely specified by a *single* parameter N which define the total $SL(2,\mathbb{R})$ spin. In the second case, a new quantum number emerges due to the fact that the corresponding dynamical system is completely integrable. The additional symmetry can be attributed to the operator q defined in (40). From the point of view of a classical dynamics this operator generates the winding of a particle around the $A-$cycles on the spectral curve. Within the string/gauge fields correspondence one expects to reproduce these properties using a description in terms of the same string propagating in different backgrounds. One is tempting to suggest that different properties of the anomalous dimensions of the two- and three-particle operators should be attributed to different properties of the background. In the case of the twist-2 the anomalous dimensions depend on integer N which in the classical system has an interpretation of the total $SL(2,\mathbb{R})$ angular momentum of the system. On the stringy side the same parameter has a natural interpretation as a string angular momentum.

In our approach we have the following correspondence

$$\begin{aligned} \text{operator} &\iff \text{Riemann surface} \\ \text{twist of the operator} &\iff \text{genus of the Riemann surface} \\ \text{calculation of the anomalous dimension} &\iff \text{quantization of the Riemann surface} \end{aligned}$$

It seems that the Riemann surfaces whose moduli (the integrals of motion of the spin chain) define the anomalous dimensions of the corresponding operators describe the $\sigma-$model solutions found in [5]. The precise relation between two approaches need to be clarified further.

As we have mentioned the quantization of the Riemann surface can be performed most effectively in terms of the Baxter equation. It is worth noting that the solution to the Baxter equation can be identified as a wave function of D0 brane [35]. Quantization conditions arise from the requirement for the wave function the D0 brane probe in the background of the Riemann surface to be well defined.

In the case of the light-cone composite operators we have to incorporate into the stringy picture a new quantum number which is parameterized by an integer n, Eq. (47), i.e. string excitation spectrum has now different sectors parameterized by this integer. The natural way to interpret these sectors is to identify n with the winding number of a closed string. The corresponding background for such scenario is offered by the Riemann surface itself with the string wrapped around the $A-$cycle. It is an interesting open question if one can interpret a momentum in WKB quantization condition (46) as a momentum of a string T-dual to the string with the winding number n.

Since the spectrum of the anomalous dimensions in QCD coincides with the spectrum of the $SL(2,\mathbb{R})$ spin chain Hamiltonian it would be interesting to explore further the symbolic relation

$$H_{\text{string}} \propto H_{\text{spin}} \tag{50}$$

where string propagates in the background determined by the Riemann surface of the spin chain. The possible link to the explanation of this relation looks as follows.It is known that hamiltonian formulation of the spin chains is closely related to the Chern-Simons (CS) theory with the inserted Wilson lines. The number of the sites in the spin chain N corresponds to the number of the Wilson lines. The gauge group in the CS theory is the symmetry group of the magnet. To get CS action from the natural AdS geometry let us remind that AdS_3 can be reformulated in terms of $SL(2,C)$ CS indeed [36]. Hence it is natural to assume that the spin chain corresponding to Regge limit describes the motion of N degrees of freedom in AdS_3 space. This issue will be discussed in more details elsewhere.

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