Universal Calabi-Yau Algebra:
Towards an Unification of Complex Geometry

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Abstract

We present a universal normal Calabi-Yau algebra suitable for constructing and classifying the infinite series of the compact complex spaces with SU(n) holonomy. This algebraic approach includes natural extensions of reflexive weight vectors to higher dimensions. It includes a 'dual' construction based on the Diophantine decomposition of invariant monomials, which provides explicit recurrence formulae for the numbers of Calabi-Yau spaces in arbitrary dimensions.

1 Introduction: an Algebraic Way to Unify Calabi-Yau Geometry

Geometrical ideas play ever-increasing rôles in the quest to unify all the fundamental interactions. They were introduced by Einstein in the formulation of general relativity, and extended to higher dimensions by Kaluza and Klein in order to include electromagnetism. This is described by the geometrical principle of gauge invariance, which is also used in the formulation of the strong and weak interactions. It is already well known that the compactification on the symmetric spaces with some isometry and holonomy groups were intensively used to extend the idea of Kaluza-Klein in the supergravity approach and then in compactification of five superstrings and M/F-theories.

The notion of the holonomy symmetry was indiced by E.Cartan for classification of all Riemannian locally symmetric spaces. The holonomy group H is one of the main characteristic of an affine connection on a manifold M. The definition of holonomy group is directly connected with parallel transport along the piece-smooth path joining two points $x \in M$ and $y \in M$. For a connected n-dimensional manifold M with Riemannian metric g and Levi-Civita connection the parralel transport along using the connection defines the isometry between the scalar products on the tangent spaces T_xM and T_yM at the points x and y. So for any point $x \in M$ one can represent the set of all linear automorphisms of the associated tangent spaces T_xM which are induced by parallel translation along x-based loop.

If a connection is locally symmetric then its holonomy group equals the local isotropy subgroup of the isometry group G. Hence, the holonomy group classification of these connections is equivalent to the classification of symmetric spaces which was done completely long ago [1] The full list of symmetric spaces is given by the theory of Lie groups through the homogeneous spaces $M = G/H$, where G is a connected group Lie acting transitively on M and H is a closed connected Lie subgroup of G, what determines the holonomy group of M. Symmetric spaces have a transitive group of isometries. The known examples of symmetric spaces are R^n, spheres S^n, CP^n etc.

Firstly, in 1955, Berger presented the classification of irreducibly acting matrix Lie groups occured as the holonomy of a torsion free affine connection. The Berger list of non-symmetric irreducible Riemannian manifolds with the list of holonomy groups H of M one can see, for example, in [4].

For H=SO(n) the holonomy principle means that there are no parallel (constant) tensor fields apart from metric and orientation. The next example $H = U(n) \subset SO(2n)$ is preserving apart from metric the complex structure J on R^{2n} which is parallel (constant) and orthogonal ($J \in SO(2n), J^2 = -1$). These manifolds with holonomy contained in U(n) are Riemannian manifolds with a complex structure J called as Kähler manifolds.

We will accent here on the infinite series of Calabi-Yau spaces with SU(n) holonomy group [3]. Following Joyce [4] it is better here to define the Calabi-Yau n-folds as a quadruple (M, J, g, Ω) where (M, J) is a complex compact n-dimensional manifold with complex structure J, g is a Kähler metrics with SU(n)-holonomy group, and Ω is a non-zero constant (parallel) $\Omega = (n, 0)$-tensor called by the holomorphic volume form.

In principle, it is enough to define the Calabi-Yau n-folds a little shorter *i.e.* a Calabi-

Yau n-fold is a compact Kähler manifold (M, J, g) of dimension n with SU(n) holonomy group. And then one can prove for Calabi-Yau n-folds the existence of constant (parallel) holomorphic $\Omega = (n, 0)$ form. More exactly, using the holonomy principle one can choose for each point $x \in M$ the complex coordinates $(z_1, ..., z_n)$ in which

$$\begin{aligned} g &= |dz_1|^2 + \ldots |dz_n|^2 \\ \omega &= \frac{i}{2}(dz_1 \Lambda d\bar{z}_1 + \ldots + dz_n \Lambda d\bar{z}_n) \\ \Omega &= dz_1 \Lambda \ldots \Lambda dz_n \end{aligned} \tag{1}$$

where the form Ω is unique up to multiplication by $\exp^{i\phi}$ for $\phi \in [0, 2\pi)$. The existence of a parallel form of type $(n, 0)$ means that the cannonical bundle $K_M := \Omega^n_M$ is flat. In other words, the Ricci curvature which for Kähler manifold is just the curvature of K_M is equal to zero. Due to Yau's proof of the Calabi conjecture one has the following: If (M,J) is a compact complex n-fold admitting Kähler metrics with trivial cannonical bundle then there exists a unique Ricci-flat metrics g in each Kähler class of M and with holonomy group $H = SU(n)$. We would like to present the possibility for algebraic solution of this infinite Calabi-Yau series of compact complex n-folds with SU(n) holonomy (see Figure 1).

2 The Arity-Dimension Structure of Universal Calabi-Yau Algebra

The starting point for our algebraic approach to the classification of Calabi-Yau spaces has therefore been the construction of 'reflexive' weight vectors $\vec{k}$, whose components specify the complex quasihomogeneous projective spaces $CP^n(k_1, k_2, ..., k_{n+1})$. These have $(n + 1)$ quasihomogeneous coordinates $x_1, ..., x_{n+1}$, which are subject to the following identification:

$$(x_1, \ldots, x_{n+1}) \sim (\lambda^{k_1} \cdot x_1, \ldots, \lambda^{k_{n+1}} \cdot x_{n+1}). \tag{2}$$

In the case of CP^n projective spaces there exists a very powerful conjecture, called Chow's theorem, that each analytic compact (closed) submanifold in CP^n can be specified by a set of polynomial equations. The set of zeroes of quasihomogeneous polynomial equations,

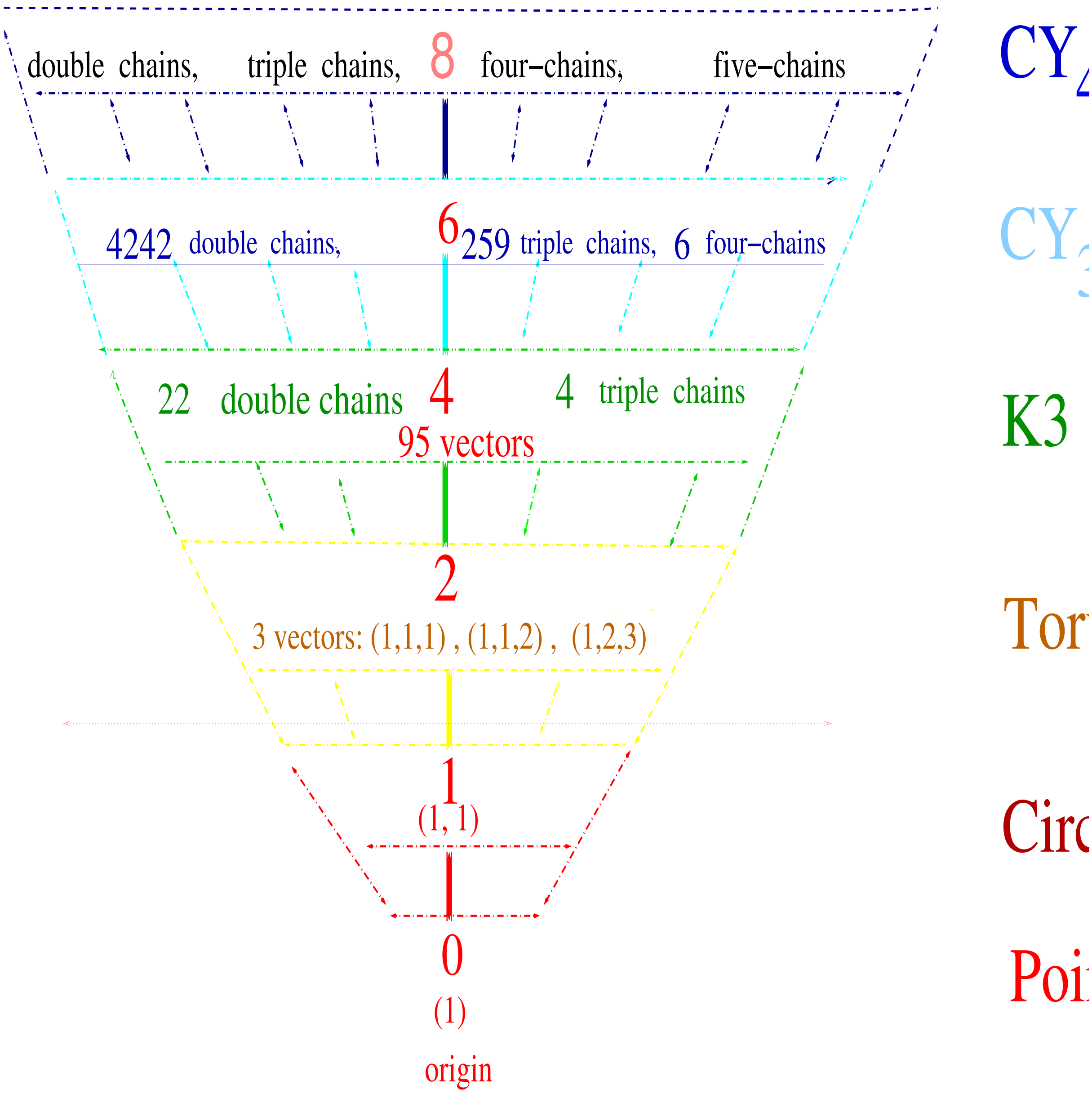

Figure 1: *The genealogical tree for Calabi-Yau n-folds.*

hereafter referred to as Calabi-Yau equations, define a projective algebraic variety in such a weighted projective space.

A d-dimensional Calabi-Yau space X_d can be given by the locus of zeroes of a transversal quasihomogeneous polynomial $\wp$ of degree $deg(\wp) = [d] : [d] = \sum_{j=1}^{n+1} k_j$ in a complex projective space $CP^n(\vec{k}) \equiv CP^n(k_1, ..., k_{n+1})$ [6]:

$$X \equiv X^{(n-1)}(k) \equiv \{\vec{x} = (x_1, ..., x_{n+1}) \in CP^n(k) | \wp(\vec{x}) = 0\}. \tag{3}$$

The general quasihomogeneous polynomial of degree $[d]$ is a linear combination

$$\wp = \sum_{\vec{\mu}_\alpha} c_{\vec{\mu}_\alpha} x^{\vec{\mu}_\alpha} \tag{4}$$

of monomials $x^{\vec{\mu}_\alpha} = x_1^{\mu_{1\alpha}} x_2^{\mu_{2\alpha}} ... x_{r+1}^{\mu_{(r+1)\alpha}}$ with the condition:

$$\vec{\mu}_\alpha \cdot \vec{k} \, = \, [d]. \tag{5}$$

This algebraic projective variety is irreducible if and only if its polynomial is irreducible. A hypersurface will be smooth for almost all choices of polynomials. To obtain Calabi-Yau d-folds one should choose reflexive weight vectors (RWVs), related to Batyrev's reflexive polyhedra or to the set of IMs. Other examples of compact complex manifolds can be obtained as the complete intersections (CICY) of such quasihomogeneous polynomial constraints:

$$X_{CICY}^{(n-r)} = \{\vec{x} = (x_1, \ldots x_{n+1}) \in CP^n \,|\, \wp_1(\vec{x}) = \ldots = \wp_r(\vec{x}) = 0\}, \tag{6}$$

where each polynomial $\wp_i$ is determined by some weight vector $\vec{k}_i$, $i = 1, \ldots, r$.

A useful technique for constructing Calabi-Yau spaces in any number of dimensions is to visualize the various possible monomials $(x_1^{\mu_1} x_2^{\mu_2} ... x_n^{\mu_{n+1}})_\alpha$ as the $m_\alpha = (\mu_1, ..., \mu_{n+1})_\alpha$ points in the Z_{n+1} integer lattice of an n-dimensional polyhedron. Using this technique, Batyrev [5] demonstrated how to associate by explicit construction a mirror polyhedron to each Calabi-Yau space. This approach also established in a very elegant way the corresponding mirror duality among Calabi-Yau spaces.

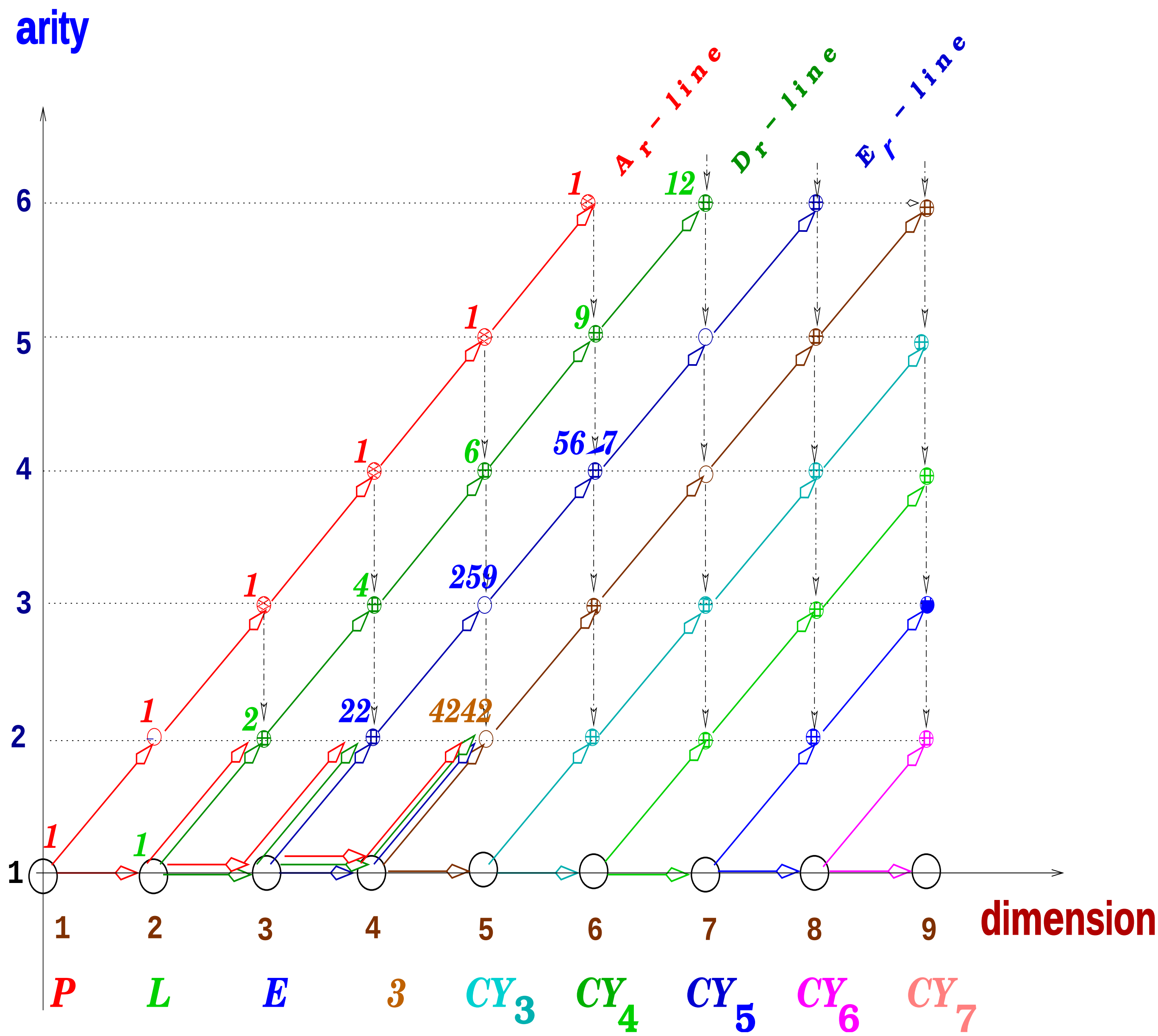

Figure 2: *The arity-dimension plot, showing the numbers of eldest vectors/chains obtained by normal extensions of RWVs, including previous results for CY_3 and lower-dimensional spaces, and new results for CY_4 and CY_5 spaces.*

The Universal Calabi-Yau Algebra (UCYA) structure of reflexive weight vectors in different dimensions depends on two integer parameters: the *arity* r of the combination operation ω_r, and the dimension n of the reflexive weight vectors (RWVs), that are connected one-to-one with Batyrev's reflexive polyhedra. These weight-vectors could be classified using the natural extensions of lower-dimensional vectors and their combination via binary, ternary, etc., operations (see Figure 2). The innovation is the introduction of a complementary algebraic approach to the construction of Calabi-Yau spaces, based on the construction of suitable monomials $\vec{\mu}$ obeying the 'duality' condition: $\vec{k} \cdot \vec{\mu}_\alpha = d$. This construction supplements the previous geometrical method related to Batyrev polyhedra, and enables one to calculate the numbers of eldest vectors, and hence chains, in arbitrary dimensions. We verify explicitly that the eldest vectors found in the two different ways agree in several instances for both CY_3 and CY_4 spaces, providing increased confidence in our results. The study of the Calabi-Yau equations and the associated hypersurfaces via the remarkable composite properties of IMs provides an alternative algebraic route to reflexive polyhedron techniques. Central rôles are played in our approach by the composite structures in lower dimensions $\leq (d-1)$ of CY d-folds, and the algebraically dual ways of expansions using weight vectors $\vec{k}$ and invariant monomials (IMs). By analogy with the Galois normal extension of fields, we term the first way of expanding weight vectors a *normal* extension, and the dual decomposition in terms of IMs we call the *Diophantine* expansion. These two expansion techniques are consistently combined in our algebraic approach, whose composition rules exhibit explicitly the internal structure of the Calabi-Yau algebra. Our method is closely connected to the well-known Cartan method for constructing Lie algebras, and reveal various structural relationships between the sets of Calabi-Yau spaces of different dimensions. We interpret our approach as revealing a 'Universal Calabi-Yau Algebra' [11] for the following reasons: 'Universal' because it may, in principle, be used to generate all Calabi-Yau manifolds of any dimension with all possible substructures, and 'Algebra' because it is based on a sequence of binary and higher n-ary operations on weight vectors and monomials.

Our objective is to construct an universal algebra [11] acting on the set of reflexive weight vectors in all dimensions, $A_n \equiv \{\mathrm{RWV}(n)\}$, and the corresponding set of invariant monomials, $\{IMs(n)\}$, which is 'dual' to A_n in the sense of (5). We note that the number of IMs is much less the full set of monomials $\vec{m}_\alpha : 1 \leq \alpha \leq \alpha_{max}$ which determine the Calabi-Yau equation. Through the IMs one can determine the highest vectors of the chains and also the full list of weight vectors in the corresponding chain. To see this, we start from the unit

IM in some dimension n and then, via a Diophantine expansion, can go on to determine the conic IMs, the cubic IMs, the quartic IMs, etc.. Similarly, one can continue this process of studying the set of IMs via the Diophantine expansions of conic IMs, of cubic IMs, etc..

The RWVs and IMs provide independent routes for constructing explicitly Calabi-Yau spaces of arbitrary dimension (including CICYs). The resulting UCYA structure of RWVs in different dimensions depends on two integer parameters, including the 'arity' r defined below, as well as the dimension n. An overview in the (n, r) plane is shown in Fig. 2, where the entries $A_n^{(r)}$ label the types of possible eldest vectors, corresponding to 'chains' of related Calabi-Yau spaces.

The algebraic-geometry realization [6, 7] of Coxeter-Dynkin diagrams provides a general characterization of the possible structures in singular limits of Calabi-Yau hypersurfaces. Thus, a deeper understanding of the origins of gauge invariance provides an additional motivation for studying string vacua via our unification of the complex geometry of $d = 1$ elliptic curves, complex tori, $K3$ manifolds, CY_3, CY_4, etc. This point is illustrated in Figs. **??**, where the points on the the first three sloping lines, labelled A_r (red), D_r (green) and E (blue), correspond to those d-folds that are characterized by the 'maximal' quotient A, D, E singularities, respectively [1]. As we discuss later in more detail, this characterization of the types of singularities is directly connected to the degrees of the associated monomials - linear, conics, cubics, quartics, etc., that appear along the corresponding sloping lines.

3 Some Results

We have presented a Universal Calabi-Yau Algebra (UCYA) which provides a two-parameter classification of $CY - d$ spaces in terms of arity and dimension. This algebra is based on the following ingredients:

- Universal composition rules
- Normal expansions and Diophantine decompositions
- Mirror symmetry

[1] To be more precise, the D line includes also A-type singularities, and the E line includes also D-type and A-type singularities.

• Singularities and link with Cartan-Lie algebras

We have shown that this algebraic approach leads us to a natural formalism for a unified description of complex geometry in all dimensions, including $K3$ spaces and Calabi-Yau d-folds for any d.

Since the description of the UCYA is based on structures with two integer parameters, the arity and dimension of the reflexive weight vectors (RWVs), we have classified the structures of CY_d spaces along the diagonal $A_r, D_r, E_r, ...$ lines in this plane. In this article we have studied only the d-folds along the first three lines, presenting new results for low d and some recurrence formulae valid for all d.

As an alternative to the Batyrev reflexive polyhedron method, we have proposed a new description of CY_d spaces based on the structures of the set of invariant monomials (IMs). We have shown that the IM approach, which is based on Diophantine decompositions, is a valuable alternative to the normal RWV expansion approach. We have demonstrated this by comparing the results of both approaches for the first three diagonal lines, A_r, D_r and E_r, in the arity-dimension plot for CY_3, CY_4 cases.

We have shown that recurrence relations for conic, cubic and quartic monomials give us the formulae for the numbers of IMs in arbitrary dimensions. This was illustrated in three cases, for CY_d spaces with $\{10\}_\Delta$, $\{9\}_\Delta$ and $\{7\}_\Delta$fibres. This confirms that, in the framework of the UCYA, the Calabi-Yau 'genome' can in principle be solved completely.

As an example of the extension procedure in the case of $K3$ manifolds, we classified [9] the 95 different possible weight vectors $\vec{k}$ in 22 binary chains generated by pairs of extended vectors, which included 90 of the total, and 4 ternary chains generated by triplets of extended vectors, which yielded 91 weight vectors of which 4 were not included in the binary chains. The one remaining $K3$ weight vector was found in a quaternary chain [9]. This algebraic construction provides a convenient way of generating all the $K3$ weight vectors, and arranging them in chains of related vectors whose overlaps yield further indirect relationships.

Moreover, our construction builds higher-dimensional Calabi-Yau spaces systematically out of lower-dimensional ones, enabling us to enumerate explicitly their fibrations. As examples, we showed previously [9, 10] how our construction reveals elliptic and $K3$ fibrations of CY_3 manifolds. Our approach may also be used to obtain the projective weight vector structure of a mirror manifold, starting from those of a given Calabi-Yau manifold.

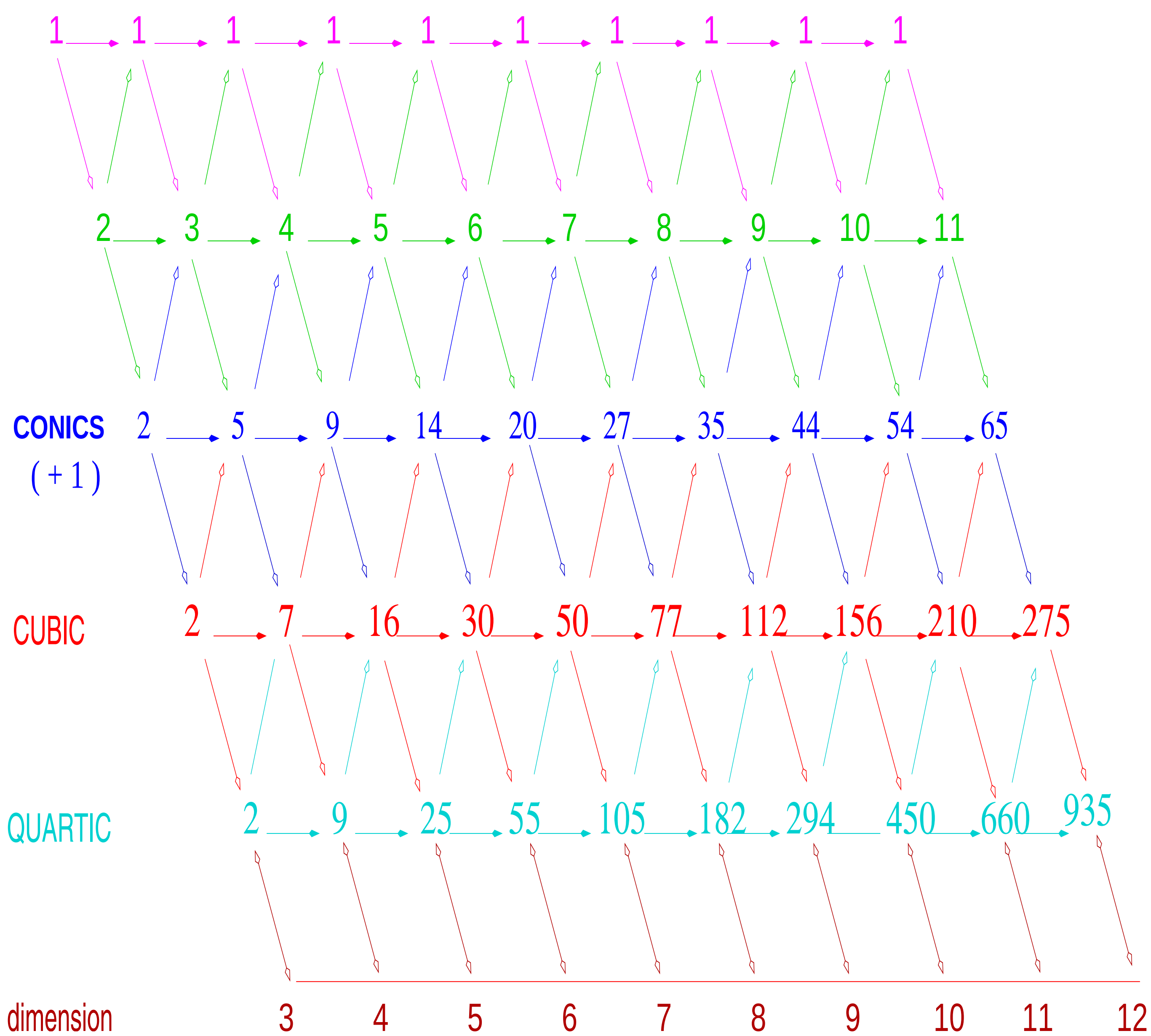

Figure 3: *Lattice illustrating recurrence relations for the numbers of conic, cubic and quartic monomials.*

One can see from the figure 3 that the IMs determine completely the fibration structures of the 22 $K3$ chains:

$$
\begin{aligned}
\{IM\}_4 \quad \mapsto \quad & \Big(1\cdot\{4\}_\triangle\Big)+\Big(\mathbf{2}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big)\\
+ \quad & \Big(2\cdot\{5\}_\triangle+1\cdot\{5\}_\square\Big)\\
+ \quad & \Big(\mathbf{4}\cdot\{\mathbf{9}\}_{\boldsymbol{\triangle}}+2\cdot\{9\}_\square\Big)\\
+ \quad & \Big(\mathbf{7}\cdot\{\mathbf{7}\}_{\boldsymbol{\triangle}}+1\cdot\{7\}_\square\Big)\\
+ \quad & \Big(1\cdot\{6\}_\square\Big)+\Big(1\cdot\{8\}_\square\Big)\\
\mapsto \quad & \{22\}
\end{aligned}
\tag{7}
$$

This expansion in terms of fibration structures is very helpful for extending these $K3$ results to more general CY_d spaces, via recurrence relations. As we show later, each of the terms $\{10,4,...\}_{\triangle,\square,...}$ in the expansion has its own recurrence relation, of which we later derive several examples, indicated in bold script: $\mathbf{2}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}$, etc., providing complete results in any number of dimensions for the numbers of CY_d spaces with these particular fibrations. A similar recurrence formula could be derived for any analogous fibration.

There are fixed types and numbers of IMs which determine the structures of the full 259 (irreducible 161) chains, and they are similar to those we already indicated for the $K3$ case, as seen, for example, in the following figure 3.

$$
\begin{aligned}
\{IM\}_5 \quad \mapsto \quad & \Big(9\cdot\{4\}_\triangle+\mathbf{4}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big)\\
+ \quad & \Big(16\cdot\{5\}_\triangle+5\cdot\{5\}_\square+1\cdot\{5\}_{\square'}\Big)\\
+ \quad & \Big(\mathbf{11}\cdot\{\mathbf{9}\}_{\boldsymbol{\triangle}}+5\cdot\{9\}_\square+1\cdot\{9\}_{\square'}\Big)\\
+ \quad & \Big(\mathbf{28}\cdot\{\mathbf{7}\}_{\boldsymbol{\triangle}}+7\cdot\{7\}_\square+1\cdot\{7\}_{Quint}\Big)\\
+ \quad & \Big(8\cdot\{6\}_\square+1\cdot\{6\}_{Quint}\Big)\\
+ \quad & \Big(6\cdot\{8\}_\square+1\cdot\{8\}_{Quint}\Big)\\
\mapsto \quad & \{161\}
\end{aligned}
\tag{8}
$$

A further reduction in the number of chains has to be considered, from the 5,607 6-dimensional 4-vector chains to 2111 independent chains. We have already mentioned that there are different types of IMs even among the cubics $\{3\}$ and quartics $\{4\}$, and the number of different conics grows monotonically with increasing dimension n. We have also already remarked that there exists a recurrence formula for all types of IMs with arbitrary dimension n, and have already discussed the reccurences of the Weierstrass IMs $\{3_W\}$ and $\{4_W\}$. The possible types of cubic $\{3\}$, quartic $\{4\}$ and double conic IMs which describe the 2111 irreducible CY_3 chains have different structures, corresponding to the different types of intersections, that we can illustrate by the following expression:

$$
\begin{aligned}
\{IM\}_6 \quad \mapsto \quad & \left(37 \cdot \{4\}_{\triangle} + \mathbf{7} \cdot \{\mathbf{10}\}_{\boldsymbol{\triangle}}\right) \\
+ \quad & \left(66 \cdot \{5\}_{\triangle} + 27 \cdot \{5\}_{\square} + 6 \cdot \{5\}_{\square'}\right) \\
+ \quad & \left(\mathbf{24} \cdot \{\mathbf{9}\}_{\boldsymbol{\triangle}} + 11 \cdot \{9\}_{\square} + 5 \cdot \{9\}_{\square'}\right) \\
+ \quad & \left(\mathbf{84} \cdot \{\mathbf{7}\}_{\boldsymbol{\triangle}} + 28 \cdot \{7\}_{\square} + 5 \cdot \{7\}_{Quint} + 1 \cdot \{7\}_{Sixt}\right) \\
+ \quad & \left(36 \cdot \{6\}_{\square} + 5 \cdot \{6\}_{Quint}\right) \\
+ \quad & \left(21 \cdot \{8\}_{\square} + 5 \cdot \{8\}_{Quint}\right) \\
\mapsto \quad & \{2111\}
\end{aligned}
\qquad (9)
$$

The recurrence relation for Calabi-Yau spaces with elliptic fibres $\{10\}_{\triangle}$ can be extended to the cases of CY_d spaces with $K3$ fibres, described by $\vec{k}_4 = (1,1,1,1)[4]$, whose algebraic equation includes the 35-point monomial and its mirror with 5 points. The IM_4 for this $K3$ space contains the four quartic monomials P_1, P_2, P_3, P_4 obeying the Diophantine equation: $(P_1+P_2+P_3+P_4)/4 = E_4$. These monomials have in addition one very important condition: $P_i - P_j$ should be divisible by 4 for each choice of $i, j = 1,2,3,4, i \neq j$. The types of different n-dimensional $\{IM\}_4$, describing the $CY_d : n = d+2 \geq 4$ spaces with $\{35\}_{\triangle}$ fibres are constructed only from the numbers 4 and 0. The number 1 will play an additional role. Therefore, similarly to the case of the third E_r line, the recurrence formulae for these IMs will be determined from the expansions of positive integer numbers in terms of four positive integers, i.e., (see Figure 4).

At last we note [9] that the lattice structure of the $K3$ projective vectors obtained by a binary construction exhibits a very interesting correspondence between the Dynkin diagrams

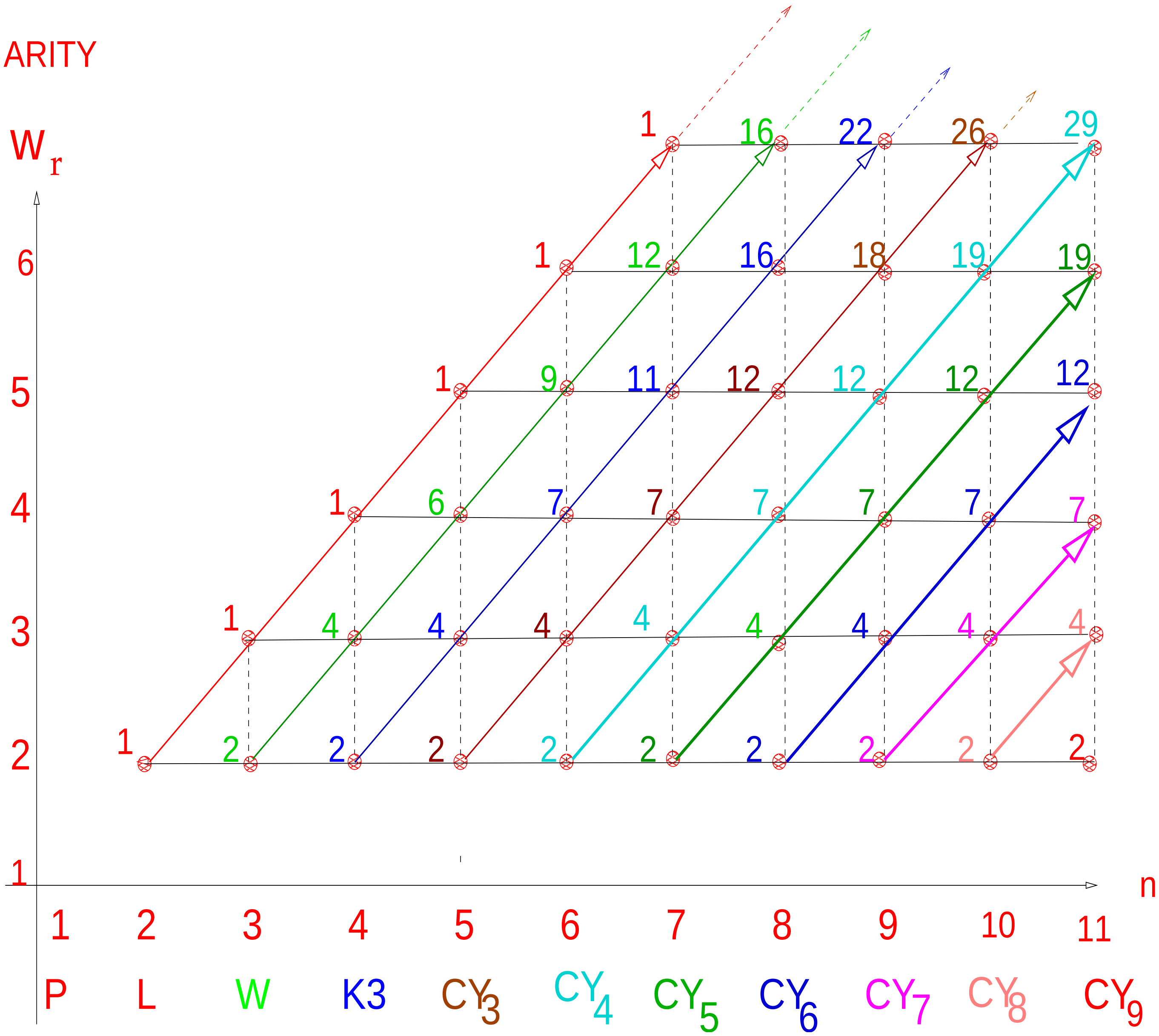

Figure 4: *The numbers of recurrences of Calabi-Yau hypersurfaces with a* $(1,...,1)_n$ *fibre are calculable along all lines* $n = r + p$ *in the arity-dimension plot.*

for Cartan-Lie groups in the A, D series and $E_{6,7,8}$ and particular reflexive weight vectors (see also Figure 5):

$$\vec{k}_1 = (1)$$

$$
\begin{aligned}
\vec{k}_2 = (1,1) \quad &\leftrightarrow \quad D_r; \\
\vec{k}_3 = (1,1,1) \quad &\leftrightarrow \quad E_6; \\
\vec{k}_3 = (1,1,2) \quad &\leftrightarrow \quad E_7; \\
\vec{k}_3 = (1,2,3) \quad &\leftrightarrow \quad E_8.
\end{aligned} \tag{10}
$$

This appearance in Calabi-Yau geometry of the A, D and E series of Cartan-Lie algebras is connected [9] with specific quotient singular structures of considered geometry like as Kleinian-Du-Val singularities C^2/Z_n.

For example, resolving the C^2/Z_n singularity gives for rational, i.e., genus zero, (-2)-curves an intersection matrix that coincides with the A_{n-1} Cartan matrix. For a general form of the C^2/G singularity, one can see [6] Any discrete subgroup of $SU(2)$ can be projected into a subgroup of $SO(3)$, and thus can be related to the finite symmetry classification of three-dimensional space. Thus, resolving the orbifold singularities yields a beautiful interrelation between the classification of finite group rotations in three-space and the ADE classification of Cartan-Lie algebras. Correspondingly, in UCYA one can see that the CY_n- polyhedra with $(n \geq 3)$ can be also constructed from n-copies of Coxeter-Dynkin diagrams.

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