The Painlevé Analysis and Solutions with Critical Points

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Abstract

The Hénon–Heiles system in the general form has been considered. In a few nonintegrable cases with the help of the Painlevé test new solutions have been found as formal Laurent or Puiseux series, depending on three parameters. One of parameters determines a location of the singularity point, other parameters determine coefficients of series. The obtained series converge in some ring. For some values of these parameters the obtained Laurent series coincide with the Laurent series of the known exact solutions.

1 THE PAINLEVÉ PROPERTY AND INTEGRABILITY

A Hamiltonian system in a $2s$–dimensional phase space is called *completely integrable* (Liouville integrable) if it possesses s independent integrals which commute with respect to the associated Poisson bracket. When this is the case, the equations of motion are (in principal, at least) separable and solutions can be obtained by the method of quadratures.

When we study some mechanical or field theory problem, we imply that time and space coordinates are real, whereas the integrability of motion equations is connected with the behavior of their solutions as functions of complex time and (in the case of the field theory) complex spatial coordinates.

S.V. Kovalevskaya was the first, who proposed [1] (see also [2, 3]) to consider time as a complex variable and to demand that solutions of the

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motion equations have to be single-valued, meromorphic functions on the whole complex (time) plane. The work of S.V. Kovalevskaya has shown the importance of application of the analytical theory of differential equations to physical problems. The essential stage of development of this theory was a classification of ordinary differential equations (ODE's) in order of types of singularities of their solutions. This classification has been made by P. Painlevé.

Let us formulate the Painlevé property for ODE's. Solutions of a system of ODE's are regarded as analytic functions, may be with isolated singular points. A singular point of a solution is said *critical* (as opposed to *non-critical*) if the solution is multivalued (single-valued) in its neighborhood and *movable* if its location depends on initial conditions[2].

Definition. A system of ODE's has ***the Painlevé property*** *if its general solution has no movable critical singular point* [4].

An arbitrary solution of such system is single-valued in the neighborhood of its singular point t_0 and can be expressed as a Laurent series with a finite number of terms with negative powers of $t - t_0$. If a system has not the Painlevé property, but, after some change of variables, the obtained system possesses this property, then the initial system is said to have *the weak Painlevé property.*

Investigations of many dynamical systems [5–7] show that a system is completely integrable only for such values of parameters, at which it has the Painlevé property[3]. At the same time the integrability of an arbitrary system with the Painlevé property has yet to be proved. There is not an algorithm for construction of the additional integral by the Painlevé analysis. It is easy to give an example of an trivially integrable system, which general solution is not a meromorphic function [9]: $H = \frac{1}{2}p^2 + f(x)$, where $f(x)$ is a polynomial, which power is not lower than five.

The Painlevé test is any algorithm designed to determine necessary conditions for a differential equation to have the Painlevé property. The original algorithm, developed by P. Painlevé and used by him to find all the second order ODE's with Painlevé property [4], is known as the α-method. The method of S.V. Kovalevskaya is not as general as the α–method, but much

[2]Solutions of a system with a time-independent Hamiltonian can have only movable singularities.

[3]Arguments, which clarify the connection between the Painlevé analysis and the existence of motion integrals, are presented in [8].

more simple[4].

In 1980, motivated by the work of S.V. Kovalevskaya [1], M.J. Ablowitz, A. Ramani and H. Segur [11] developed a new algorithm of the Painlevé test for ODE's. The remarkable property of this test is that it can be checked in a finite number of steps. They also were the first to point out the connection between the nonlinear partial differential equations (PDE's), which are soluble by the inverse scattering transform method, and ODE's with the Painlevé property. Subsequently the Painlevé property for PDE was defined and the corresponding Painlevé test (the WTC procedure) was constructed [12, 13] (see also [10, 14, 15]).

The algorithm for finding special solutions for ODE system in the form of a finite expansion in powers of unknown function $\varphi(t-t_0)$ has been constructed in [16]. The function $\varphi(t-t_0)$ and coefficients have to satisfy some system of ODE, often more simple than an initial one. This method has been used [17] to construct exact solutions for certain nonintegrable systems of ODE's. With the help of the perturbative Painleve test [14] four-parameter generalization of an exact three-parameter solution of the Bianchi IX cosmological model (Mixmaster university) has been constructed [18].

The aim of this paper is to find new special solutions for the generalized Hénon–Heiles system using the Painlevé test. We obtain solutions as formal Laurent or Puiseux series and find domains of their convergence.

2 THE HÉNON–HEILES HAMILTONIAN

In the 1960s, the models of star motion in an axial-symmetric and time-independent potential were actively studied [19, 20]. Due to the symmetry of the potential the considered system is equivalent to two-dimensional one. However, for many polynomial potentials the obtained system has not the second integral as a polynomial function. To answer the question about the existence of the third integral Hénon and Heiles [20] considered the behavior of numerically integrated trajectories. Emphasizing that their choice of potential does not proceed from experimental data, they have proposed the Hamiltonian

$$H=\frac{1}{2}\Big(x_t^2+y_t^2+x^2+y^2\Big)+x^2y-\frac{1}{3}y^3, \qquad (1)$$

[4]Different variants of the Painlevé test are compared in [10, R. Conte paper]

because: on the one hand, it is analytically simple; this makes the numerical computations of trajectories easy; on the other hand, it is sufficiently complicated to give trajectories which are far from trivial. Indeed, for low energies trajectories (numerically integrated) always lay on well-defined two-dimensional surfaces. On the other hand, for high energies many of these integral surfaces are destroyed.

The generalized Hénon–Heiles system is described by the Hamiltonian:

$$H = \frac{1}{2}\Big(x_t^2 + y_t^2 + \lambda x^2 + y^2\Big) + x^2 y - \frac{C}{3}\, y^3 \qquad (1')$$

and the corresponding system of the motion equations:

$$\left\{ \begin{array}{lcl} x_{tt} & = & -\lambda x - 2xy, \\ y_{tt} & = & -y - x^2 + C y^2, \end{array} \right. \qquad (2)$$

where $x_{tt} \equiv \frac{d^2x}{dt^2}$ and $y_{tt} \equiv \frac{d^2y}{dt^2}$, λ and C are numerical parameters.

Due to the Painlevé analysis the following integrable cases of (2) have been found:

(i) $C = -1$, $\lambda = 1$,
(ii) $C = -6$, λ is an arbitrary number,
(iii) $C = -16$, $\lambda = \frac{1}{16}$.

The Hénon–Heiles system is a model, not only actively investigated by various mathematical methods[5], but also widely used in physics, in particular, in gravitation [22, 23].

3 NONINTEGRABLE CASES

The general solutions of the Hénon–Heiles system are known only in integrable cases [24], in other cases not only four-, but even three-parameter exact solutions yet have to be found.

The Hénon–Heiles system as a system of two second order ODE's is equivalent to the fourth order equation[6]:

$$\begin{array}{rcl} y_{tttt} & = & (2C-8)y_{tt}y - (4\lambda+1)y_{tt} + 2(C+1)y_t^2 + \\ & + & \dfrac{20C}{3}y^3 + (4C\lambda - 6)y^2 - 4\lambda y - 4H, \end{array} \qquad (3)$$

[5] The history of study of the generalized Hénon–Heiles system see in [21].

[6] For given $y(t)$ the function $x^2(t)$ is a solution of a linear equation. System (2) is invariant to exchange x to $-x$.

where H is the energy of the system.

To find a special solution of the given equation one can assume that y satisfies some more simple equation. For example, the well-known solutions in terms of the Weierstrass elliptic functions [25] satisfy the following first-order differential equation:

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^2 + \tilde{\mathcal{C}}y + \tilde{\mathcal{D}}, \tag{4}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are some constants. $\tilde{\mathcal{D}}$ is proportional to energy H (arbitrary parameter), therefore, solutions of (4) are two-parameter ones.

E.I. Timoshkova [26] generalized equation (4):

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^2 + \tilde{\mathcal{C}}y + \tilde{\mathcal{D}} + \tilde{\mathcal{G}}y^{5/2} + \tilde{\mathcal{E}}y^{3/2} \tag{5}$$

and found new one-parameter sets of solutions of the Hénon–Heiles system in nonintegrable cases ($C = -\frac{4}{3}$ or $C = -\frac{16}{5}$, λ is an arbitrary number). These solutions (i.e. solutions with $\tilde{\mathcal{G}} \neq 0$ or $\tilde{\mathcal{E}} \neq 0$) are derived only at $\tilde{\mathcal{D}} = 0$, therefore, substitution $y = \varrho^2$ gives:

$$\varrho_t^2 = \frac{1}{4}\Big(\tilde{\mathcal{A}}\varrho^4 + \tilde{\mathcal{G}}\varrho^3 + \tilde{\mathcal{B}}\varrho^2 + \tilde{\mathcal{E}}\varrho + \tilde{\mathcal{C}}\Big). \tag{6}$$

The general solution of (6) has one arbitrary parameter and can be expressed in elliptic functions.

In this paper I analyze system (2) at $C = -\frac{16}{5}$ or $C = -\frac{4}{3}$ and arbitrary value of λ. The case $C = -\frac{16}{5}$ and $\lambda = \frac{1}{9}$ has been analyzed in details in my paper [21]. In this case some solutions of (6) can be expressed in elementary trigonometric functions.

4 RESULTS OF THE PAINLEVÉ TEST

We assume that the behavior of solutions in a sufficiently small neighborhood of the singularity is algebraic, it means that x and y tend to infinity as some powers of $t - t_0$:

$$x = a_\alpha (t - t_0)^\alpha \qquad \text{and} \qquad y = b_\beta (t - t_0)^\beta, \tag{7}$$

where α, β, a_α and b_β are some constants. We assume that real parts of α and β are less then zero, and, of course, $a_\alpha \neq 0$ and $b_\beta \neq 0$.

If α and β are integer numbers, then substituting

$$x = a_\alpha (t-t_0)^\alpha + \sum_{k=1+\alpha}^{N_{max}} a_k (t-t_0)^k, \qquad y = b_\beta (t-t_0)^\beta + \sum_{k=1+\beta}^{N_{max}} b_k (t-t_0)^k$$

one can transform the ODE system into a set of linear algebraic systems in coefficients a_k and b_k. With the help of some computer algebra system, for example, the system **REDUCE** or **MATEMATICA**, these systems can be solved step by step and solutions can be automatically found with any accuracy. But previously one has to determine values of constants α, β, a_α and b_β and to analyze systems with zero determinants. Such systems correspond to new arbitrary constants or have no solutions. Powers at which new arbitrary constants enter are called *resonances*. The Painlevé test gives all information about possible dominant behaviors and resonances. Moreover, the results of the Painlevé analysis point out cases, in which it is useful to include into expansion terms with fractional powers of $t - t_0$.

For the generalized Hénon-Heiles system there exist two possible dominant behaviors and resonance structures [7, 27]:

Case 1:	*Case 2*: $(\beta < \Re e(\alpha) < 0)$
$\alpha = -2$,	$\alpha = \frac{1 \pm \sqrt{1-48/C}}{2}$,
$\beta = -2$,	$\beta = -2$,
$a_\alpha = \pm 3\sqrt{2+C}$,	$a_\alpha = c_1$ **(arbitrary)**,
$b_\beta = -3$,	$b_\beta = \frac{6}{C}$,
$r = -1,\ 6,\ \frac{5}{2} - \frac{\sqrt{1-24(1+C)}}{2},\ \frac{5}{2} + \frac{\sqrt{1-24(1+C)}}{2}.$	$r = -1,\ 0,\ 6,\ \mp\sqrt{1-\frac{48}{C}}.$

In the Table the values of r denote resonances: $r = -1$ corresponds to arbitrary parameter t_0; $r = 0$ (in the *Case 2*) corresponds to arbitrary parameter c_1. Other values of r determine powers of t, to be exact, $t^{\alpha+r}$ for x and $t^{\beta+r}$ for y, at which new arbitrary parameters enter (as solutions of systems with zero determinants).

For integrability of system (2) all values of α and r have to be integer (or rational) and all systems with zero determinants have to have solutions at all values of included in them free parameters. It is possible only in the cases (i)—(iii). Those values of C, at which α and r are integer (or rational) numbers either only in the *Case 1* or only in the *Case 2*, are of interest for search of special solutions.

5 NEW SOLUTIONS

5.1 Finding of solutions in the form of formal Laurent series

Let us consider the Hénon–Heiles system with $C = -\frac{16}{5}$ to find special solutions. In the *Case 2* $\alpha = -\frac{3}{2}$ and $r = -1,\ 0,\ 4,\ 6$, hence, in the neighborhood of the singular point t_0 we have to seek x in such form that x^2 can be expand into Laurent series, beginning from $(t-t_0)^{-3}$. Let $t_0 = 0$, substituting

$$x = \sqrt{t}\left(c_1 t^{-2} + \sum_{k=-1}^{\infty} a_k t^k\right) \qquad \text{and} \qquad y = -\frac{15}{8}t^{-2} + \sum_{k=-1}^{\infty} b_k t^k$$

in (2), we obtain the following sequence of linear system in a_k and b_k:

$$\left\{\begin{array}{l} \Big(k^2-4\Big)a_k + 2c_1 b_k = -\lambda a_{k-2} - 2\displaystyle\sum_{j=-1}^{k-1} a_j b_{k-j-2}, \\ \Big((k-1)k-12\Big)b_k = -\,b_{k-2} - \displaystyle\sum_{j=-2}^{k-1} a_j a_{k-j-3} - \frac{16}{5}\sum_{j=-1}^{k-1} b_j b_{k-j-2}. \end{array}\right. \qquad (8)$$

If $k = 2$ or $k = 4$, then the determinant of (8) is equal to zero. To determine a_2 and b_2 we have the following system:

$$\left\{\begin{array}{l} c_1\Big(557056c_1^8 + (15552000\lambda - 4860000)c_1^4 + 864000000b_2 + \\ +\,108000000\lambda^2 - 67500000\lambda + 10546875\Big) = 0, \\ 818176c_1^8 + \Big(15660000\lambda - 4893750\Big)c_1^4 - \\ -\,810000000b_2 - 6328125 = 0. \end{array}\right. \qquad (9)$$

As one can see this system does not include terms, which are proportional to a_2, hence, a_2 is an arbitrary parameter (a constant of integration). We discard the solution with $c_1 = 0$ and obtain that system (9) has solutions if and only if:

$$c_1^4 = \frac{1125(4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 1680\lambda + 525)}{167552} \qquad (10.1)$$

or

$$\tilde{c}_1^4 = \frac{1125(-4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 1680\lambda + 525)}{167552}. \qquad (10.2)$$

We obtain new constant of integration a_2, but we must fix c_1, so number of constants of integration is equal to 2. It is easy to verify that b_4 is an arbitrary parameter, because the corresponding system is equivalent to one linear equation. So, using Painlevé test, we obtain solutions which depend on three parameters, namely t_0, a_2 and b_4. For given λ we obtain four different three-parameter solutions. Each solution generalizes exact one-parameter solution in terms of elliptic functions.

When a formal series is obtained the question about its convergence arises. The convergence of psi-series solutions of the generalized Hénon–Heiles system on some real time interval has been proved in [27]. For Laurent series solutions it is easy to find conditions, at which the obtained series converge at $0 < |t| \leqslant 1 - \varepsilon$, where ε is any positive number. Our series converge in the above-mentioned ring, if $\exists N$ such that $\forall n > N \;\; |a_n| \leqslant M$ and $|b_n| \leqslant M$. Let $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for all $-1 < n < k$, then from (8) we obtain:

$$|a_k| \leqslant \frac{2M(k+1) + |\lambda| + 2|c_1|}{|k^2 - 4|}M, \qquad |b_k| \leqslant \frac{21Mk + 26M + 5}{5|k^2 - k - 12|}M. \qquad (11)$$

It is easy to see that there exists such N that if $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n \leqslant N$, then $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n < \infty$. So one can prove the convergence, analyzing values of a finite number of the first coefficients of series.

For $C = -\frac{4}{3}$ we obtain the analogical situation. Each of four one-parameter periodic solutions, which have been found in [26], can be generalized to three-parameter Laurent series solutions.

Let us consider solutions of equation (4). For some values of C two-parameter solutions in terms of the Weierstrass elliptic functions can be generalized. For example, if $C = -\frac{9}{8}$, then some resonances are half-integer. Substituting

$$x = \sum_{k=-4}^{\infty} \tilde{a}_k t^{k/2} \qquad \text{and} \qquad y = \sum_{k=-4}^{\infty} \tilde{b}_k t^{k/2}$$

in (2) , we obtain three-parameter solutions as formal Puiseux series for any λ.

CONCLUSION

Using the Painlevé analysis one can not only find integrable cases of dynamical systems, but also construct special solutions in nonintegrable cases.

We have found the special solutions of the generalized Hénon–Heiles system with $C = -\frac{16}{5}$ and $C = -\frac{4}{3}$ as formal Laurent series, depending on three parameters. For some values of two parameters the obtained solutions coincide with the known exact periodic solutions. At $C = -\frac{9}{8}$ two-parameter solutions in terms of the Weierstrass elliptic functions can be generalized to three-parameter ones. New solutions have been found as formal series. In [27] it has been proved that psi-series solutions have nonzero domain of convergence, thereby the obtained formal solutions are actual solutions.

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