Gauge field systems in higher dimensions

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Abstract

Gauge field (Yang–Mills) systems supporting finite action/energy topologically stable solutions in higher dimensions are reviewed. The cases of gauged Higgs and gauged Sigma systems are emphasised.

1 Gauge fields on $\mathbf{M}_{2n}$

Using the notation $F(2) = F_{\mu\nu}$ for the 2-form Yang–Mills (YM) curvature, the $2p$-form YM tensor

$$F(2p) = F(2) \wedge F(2) \wedge ... \wedge F(2) \ , \quad p - \mathrm{times} \tag{1}$$

is a p fold totally antisymmetrised product of the 2-form curvature.

In $2n$ dimensions, partitionaing n as $n = p + q$, the Hodge dual of the $2q$-form field $F(2q)$, namely $({}^\star F(2q))(2p)$, is a $2p$-form.

Starting from the inequality

$$\mathrm{Tr}[F(2p) - \kappa \ \ {}^\star F(2q)]^2 \geq 0 \ , \tag{2}$$

it follows that

$$\mathrm{Tr}[F(2p)^2 + \kappa^2 \ F(2q)^2] \geq 2\kappa \, \mathcal{C}_n \ , \tag{3}$$

where $\mathcal{C}_n$ is the n-th Chern-Pontryagin density. In (2) and (3), the constant κ has the dimension of length to the power of $(p - q)$.

The (generalised) YM systems in (even) higher dimensions are characterised by Lagrangians defined by the densities on the *left hand side* of (3). When in particular $p = q$, then these systems are conformally invariant.

The inequality (3) presents a topological lower bound which guarantees that finite action solutions to the Euler–Lagrange equations exist. Of particular interest are solutions to first order self-duality equations which solve the second order Euler–Lagrange equations, when (3) can be saturated.

For $\mathbf{M}_{2n} = \mathbf{R}_{2n}$, the self–duality equations support nontrivial solutions only if $q = p$,

$$F(2p) \ = \ {}^\star F(2p) \ . \tag{4}$$

For $p = 1$, i.e. in four Euclidean dimensions, (4) is the usual YM selfduality euqations supporting instanton solutions. Of these, the spherically symmetric [1] and axially symmetric[2] instantons are the ones relevant to us here. This is because for $p \geq 2$, i.e. in dimensions eight and higher, only sphericaly symmetric [3] and axially symmetric [4] solutions can be constructed, because in these dimensions (4) are overdetermined [5].

In the $r \gg 1$ region, all these 'instanton' fields on $\mathbf{R}_{2n}$, whether self–dual or not, asymptotically behave as pure–gauge

$$A \to g d g^{-1}$$

In addition to (regular) instantons, the Euler–Lagrange equations satisfy (singular) 'Meron' solutions [8] in $4p$ dimensions when the Lagrangian in (3) is conformally invariant, generalising the Meron solutions [9] of the usual YM system for $p = 1$.

For $\mathbf{M}_{2n} = G/H$, namely on compact coset spaces, the self–duality equations support nontrivial solutions for all p and q,

$$F(2p) \ = \ \kappa \ {}^\star F(2q) \tag{5}$$

where the constant κ is some power of the 'radius' of the (compact) space. The simplest examples are $\mathbf{M}_{2n} = S^{2n}$, the $2n$-spheres [6], and $\mathbf{M}_{2n} = \mathbf{CP}^n$, the complex projective spaces [7].

2 Higgs models on $\mathbf{R}_d$

Higgs fields have the same dimensions as gauge connections and appear as the extra components of the latter under dimensional reduction, when the extra dimension is a compact symmetric space. In general one can employ a linear combination of inequalities (3), for all $p \leq d/4$ and $q \leq d/4$. Restricting, for simplicity to the $4p$ dimensional conformal invariant systems in (3), i.e. to $p = q = d/4$, the descent over the compact space K^{4p-d} is described by

$$\int_{\mathbf{R}_d \times K^{4p-d}} F(2p)^2 \ \geq \int_{\mathbf{R}_d \times K^{4p-d}} \mathcal{C}_{2p} \ . \tag{6}$$

Imposing the symmetry appropriate to K^{4p-d} on the gauge fields results in the breaking of the original gauge group g to, say, the residual gauge group $\bar{g}$ for the fields on $\mathbf{R}_d$. Performing then the compact integration over K^{4p-d} results in the Lagrangian $\mathcal{L}[\mathcal{A},\phi]$ of the residual Higgs model on $\mathbf{R}_d$. $\mathcal{A}$ here is the connection taking values in the algebra of $\bar{g}$ and ϕ is the Higgs multiplet whose structure under $\bar{g}$ depends on the detailed choice of K^{4p-d}, implying the following gauge transformations

$$\mathcal{A} \to \bar{g}\mathcal{A}\bar{g}^{-1} + \bar{g}d\bar{g}^{-1}$$

and depending on the choice of K^{4p-d},

$$\phi \to \bar{g}\phi\bar{g}^{-1} \quad , \quad \text{or} \quad , \quad \phi \to \bar{g}\phi \quad , \quad \text{etc.}$$

The inequality (3) leads, after this dimensional descent, to

$$\int_{\mathbf{R}_d} \mathcal{L}[\mathcal{A},\phi] \ \geq \ \int_{\mathbf{R}_d} \nabla.\mathbf{\Omega}[\mathcal{A},\phi] \tag{7}$$

$$= \int_{\mathbf{\Sigma^{d-1}}} \mathbf{\Omega}[\mathcal{A},\phi] \ , \tag{8}$$

where $\mathcal{L}[\mathcal{A},\phi] = \mathcal{L}[\mathcal{F},\mathcal{D}\phi,|\phi|^2,\eta^2]$ is the residual Lagrangian in terms of the residual gauge connection $\mathcal{A}$ and its curvature $\mathcal{F}$, the Higgs fields ϕ and its covariant derivative $\mathcal{D}\phi$ and the inverse of the compactification 'radius' η. The latter is simply the VEV of the Higgs field, seen clearly from the typical form of the components of the curvature F on the extra (compact) space K^{4p-d}

$$F|_{K^{4p-d}} \ \sim \ (\eta^2 - |\phi|^2) \otimes \Sigma \quad \Rightarrow \quad \lim_{r\to\infty} |\phi|^2 = \eta^2 \tag{9}$$

where Σ are spin-matrices/Clebsch-Gordan coefficients. ' Note that the integrand on the right hand side of (7) is a total divergence [10] just like the p-th Chern-Pontryagin density from which it has descended. It is reasonable to call the density $\mathbf{\Omega}[\mathcal{A},\phi]$ the residual Chern-Simons density.

2.1 Some properties of $\mathbf{\Omega}[\mathcal{A},\phi]$ and $\mathcal{L}[\mathcal{A},\phi]$

- When $4p-1$ is **odd**, $\mathbf{\Omega}[\mathcal{A},\phi]$ is **gauge invariant**
- When $4p-1$ is **even**, $\mathbf{\Omega}$ has the form

  $$\mathbf{\Omega}[\mathcal{A},\phi] = \mathbf{\Omega_{CS}}[\mathcal{A}] + \mathbf{\Omega_0}[\mathcal{F},\mathcal{D}\phi,|\phi|^{\mathbf{2}},\eta^{\mathbf{2}}]$$

  where $\mathbf{\Omega_0}$ is a gauge invariant density and $\mathbf{\Omega_{CS}}$ is the gauge variant Chern-Simons density in those (even) dimensions. Hence $\mathbf{\Omega}$ is **gauge variant**.
- When $4p-1$ is **even** the residual gauge field consists of two distinct, say $n\times n$, gauge connections in the algebra of the residual gauge group $\bar{g}$, say $n\times n$ (anti)Hermitian fields $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$, and, an Abelian connection a_μ. The Higgs field then consists of the $n\times n$ complex array φ.

In the special case where $K^{4p-d} = S^{4p-d}$, i.e. when the compact extra space is a sphere, both the residual Lagrangian and the residual Chern-Simons densities can be expressed in terms of the 'chirally symmetric' multiplets

$$(\mathcal{A}, \mathcal{B}, \varphi) \quad \mapsto \quad A = \left[\begin{array}{cc} \mathcal{A}^{(+)} & 0 \\ 0 & \mathcal{A}^{(-)} \end{array} \right] \quad , \quad \Phi = \left[\begin{array}{cc} 0 & \varphi \\ \varphi^{\dagger} & 0 \end{array} \right] .$$

With the appropriate choice of n the residual connection is in the algebra of $SO(d)$. This is the main case of interest and leads to the construction of Wu-Yang [11] fields in all dimensions, and their associated singular connections in the Dirac gauge.

With any other choice of the compact space the residual theory will not exhibit this property of chiral symmetry, for example with $p = 2$ and $d = 4$

- $K^4 = S^4$ leads to a $SO(4) \times U(1)$ residual connection which is **chirally symmetric**
- $K^4 = \mathbf{CP}^2$ leads to $SU_1(2) \times SU_2(2) \times U(1)$ residual connection which is **not chirally symmetric**.

- The detailed features of the Higgs (self-interaction) potential will depend on the mode of descent, namely on the choice of the compact space K^{4p-d}. For example for $K = S^2 \times S^2 \times ...$, each S^2 will be characterised by its own radius, and in this example there will arise the vacuum expectation values η_1, η_2,... each via the generic form (9) with the appropriate Σ's for each η. For a direct product compact space consisting of two 2-spheres, the residual Lagrangian will typically be of the form

$$\mathcal{L}_{res} = \mathcal{L}_0 + (\eta_2^2)^{2p} \ ,$$

in which $\mathcal{L}_0$ vanishes asymptotically by the finite action/energy condition. Clearly the 'cosmological constant' term $(\eta_2^2)^{2p}$ causes the action/energy to diverge, thus eliminating the usefulness of models arising from such modes of descent.

- Saturation of the Bogomol'nyi inequality (7)-(8) is impossible in the generic case because the resulting Bogomol'nyi equations are *overdetermined* [5]. The only examples for which the Bogomol'nyi equations have nontrivial solutions pertain to:

  - in $d = 2$, for models characterised by all possible values of p and linear combinations thereof. These are the two dimensional generalised Abelian Higgs models [12] whose solutions are intimately related to the axially symmetric self-dual solutions of the conformal invariant YM systems in $4p$ dimensions [4].
  - in $d = 3$, and *only* for the model characterised by $p = 1$, i.e. for the Georgi-Glashow model in the Prasad-Sommerfield limit.

3 Gauged sigma models

In contrast to Higgs fields, sigma model fields (which take their values on a compact coset space) are dimensionless and cannot be identified as components of a connection in higher dimensions. Their gauging prescription therefore is not as constrained (and hence unique) as that for Higgs models.

Perhaps the most important difference between Higgs models and sigma models is the fact that the equations of motion of sigma models support solitons *in the absence of a gauge field*, which is impossible for a Higgs model, *cf* there are no solitons of (conventional) Goldstone models.

Using the criterion of gauging to be that the action/energy of the gauged system is bounded from below by a topological charge, sigma models can be categorised in two main families, each characterised by the topological charge. On the one hand we have the gauged Grassmannian models with the Pontryagin charge of the gauge field as the topological charge. On the other we have the d dimensional $O(d+1)$ models whose fields take values on S^d, with action/energy bounded by the winding number, or the *degree of the map*. The latter is of greater interest since its solitons can survive the gauge decoupling limit, while in the case of the former the Pontryagin charge disappears under gauge decoupling.

3.1 Gauged Grassmannian models

Grassmannian co-ordinates (fields) Z have $m \times n$ complex values subject to the $n \times n$ constraint

$$Z^\dagger Z = 1\!\!1 \ , \tag{10}$$

while the $m \times m$ symbol $(1\!\!1 - ZZ^\dagger)$ is a projection operator. These systems, typified by the kinetic terms

$$\mathrm{Tr}|\tilde{D}_\mu Z|^2 \stackrel{def}{=} \mathrm{Tr}\tilde{D}_\mu Z^\dagger \tilde{D}_\mu Z \ \ , \ \ \mathrm{Tr}|\tilde{D}_\mu Z^\dagger \wedge \tilde{D}_\nu Z|^2 \ \ , \ \ \mathrm{Tr}|\tilde{D}_\mu Z^\dagger \wedge \tilde{D}_\nu Z \wedge \tilde{D}_\rho Z^\dagger|^2 \ \ , \ \ \mathrm{etc}\ , \tag{11}$$

possess an inherent (local gauge) freedom in their definitions. The covariant derivatives in (11) are defined in terms of a *composite connection* B_μ

$$\tilde{D}_\mu Z = \partial_\mu - ZB_\mu \quad , \quad B_\mu = Z^\dagger \partial_\mu Z \ . \tag{12}$$

The simplest prescription of gauging (and the only one known to this speaker) is **replacing the composite gauge connection B_μ describing the inherent gauge freedom of the system, by a dynamical gauge connection A_μ with nonvanishing curvature**[13, 14]. Thus one replaces the covariant derivative $\tilde{D}_\mu$ in (11) and (12) by

$$D_\mu Z = \partial_\mu - ZA_\mu \quad , \quad [D_\mu, D_\nu]Z = -ZF_{\mu\nu} \ . \tag{13}$$

To examine the topological lower bound on a typical gauged Grassmannian model, consider the simplest case in dimensions $d > 2$ (in which case the Grassmannian $\mathbf{CP}^1$ coincides with S^2), namely the $SO_\pm(4)$ gauged system in $d = 4$. There, the Bogomol'nyi inequality

$$\mathrm{Tr}[D_{[\mu}Z^\dagger D_{\nu]}Z \ - \ {}^\star F_{\mu\nu}]^2 \geq 0 \ , \tag{14}$$

results in

$$\mathrm{Tr}(|F_{\mu\nu}|^2 + |D_{[\mu}Z^\dagger D_{\nu]}Z|^2) \geq \mathrm{Tr}F_{\mu\nu}\ {}^\star F_{\mu\nu} + \partial_\mu \mathrm{Tr}^\star F_{\mu\nu}D_\nu Z \ . \tag{15}$$

The second term in (15) decays too fast and contributes nothing to the surface integral, so the only contribution comes from the leading term, namely the Pontryagin charge. This is exactly the same as in the Higgs models. Thus, at least in the context of gauging a Grassmannian model with the given prescription, the soliton does not survive the gauge decoupling limit, thus limiting its interest.

3.2 $SO(N)$ gauged $O(d+1)$ models on $\mathbf{R_d}$ $(2 \leq N \leq d)$

The d dimensional $O(d+1)$ models, the best known example of which is the Skyrme models [15] (for $d = 3$), are described by the S^d valued fields χ^a $(a = 1, 2, .., d+1)$, subject to

$$|\chi^a|^2 = 1 \ .$$

Partitioning the label a as $a = \alpha, A$, with $\alpha = 1, 2, .., N$ and $A = N+1, N+2, .., d+1$, the gauging prescription is stated as

$$\begin{aligned} \chi^\alpha \to (g\chi)^\alpha \quad &\Rightarrow \quad D_\mu \chi^\alpha \stackrel{def}{=} \partial_\mu \chi^\alpha + A_\mu \chi^\alpha \to (gD_\mu\chi)^\alpha \quad ; \quad A_\mu^{[\alpha\beta]} \in so(N) \\ \chi^A \to \chi^A \quad &\Rightarrow \quad D_\mu \chi^A \stackrel{def}{=} \partial_\mu \chi^A \to D_\mu \chi^A \ , \end{aligned}$$

such that the ungauged components of the field labeled by A include the component $A = d+1$, on the understanding that one requires the usual asymptotic conditions

$$\lim_{r\to 0} \chi^{d+1} = -1 \qquad , \qquad \lim_{r\to\infty} \chi^{d+1} = +1 \ . \tag{16}$$

In contrast to the Grassmannian case above, it is obvious that going to the gauge decoupling limit here, i.e. setting $A_\mu = 0$, reverts to the ungauged system which supports a soliton. Thus the topological charge of both the gauged and the ungauged systems must be equal. This criterion was first used in [16] for $d = 3$ and later applied in [17] for $d = 2$.

The definition [18] of the topological charge of the gauged $O(d+1)$ model proceeds as follows. Up to a numerical factor of the angular volume, the winding number density is

$$\varrho_0 \sim \varepsilon_{\mu_1\mu_2...\mu_d}\varepsilon^{a_1a_2...a_da_{d+1}}\partial_{\mu_1}\chi^{a_1}\partial_{\mu_2}\chi^{a_2}...\partial_{\mu_d}\chi^{a_d}\chi^{d+1} \ . \tag{17}$$

ϱ_0 being *gauge variant*, it is unsatisfactory as a definition for the topological charge density, which must be *gauge invariant*.

On the other hand, the volume integral of the gauge covariantised version of (17),

$$\varrho_G \sim \varepsilon_{\mu_1\mu_2...\mu_d}\varepsilon^{a_1a_2...a_da_{d+1}} D_{\mu_1}\chi^{a_1} D_{\mu_2}\chi^{a_2}...D_{\mu_d}\chi^{a_d}\chi^{d+1} \ , \tag{18}$$

is not an integer, and hence also not suitable as a definition for the topological charge density.

The relation between ϱ_0 and ϱ_G on the other hand, offers the resolution to this problem. Writing

$$\varrho_G = \varrho_0 + \mathcal{W}[A_\mu^{[\alpha\beta]};\chi^a] \ , \tag{19}$$

the quantity denoted by $\mathcal{W}$ is manifestly *gauge variant*. It was found however that for $SO(d)$ gauging, with $d = 2, 3, 4$ [18], and for $SO(N)$ gauging with ($N = 2,\ d = 3$) [19], and ($N = 3,\ 2;\ d = 4$), the density $\mathcal{W}$ could be decomposed in two parts

$$\mathcal{W}[A_\mu^{[\alpha\beta]};\chi^a] = \mathcal{V}[F_{\mu\nu}^{[\alpha\beta]}, D_\mu\chi^\alpha, \chi^A] + \partial_\mu\Omega_\mu[A_\mu^{[\alpha\beta]};\chi^a] \tag{20}$$

such that the first term $\mathcal{V}$ is *gauge invariant* and the second, which is necessarily *gauge variant* is a total divergence.

What is crucial now is that the surface integral of the density Ω_μ vanishes by virtue of finite action/energy conditions imposed on the corresponding Lagrangian density.

For all **odd** d this is immediatey true, while in all **even** d the leading term in the density Ω_μ always turns out to be Ω_{CS}, the Chern-Simons density of the given dimensions, whose surface integral certainly does not vanish. But this can be remedied by redefining Ω_μ by subtracting from it Ω_{CS}, and preserving the equality (20) by adding the divergence of this Chern-Simons density (i.e. the Pontryagin density) to $\mathcal{V}$, i.e. for even d (20) is replaced by

$$\begin{aligned}
\tilde{\mathcal{W}}[A_\mu^{[\alpha\beta]};\chi^a] &= \tilde{\mathcal{V}}[F_{\mu\nu}^{[\alpha\beta]}, D_\mu\chi^\alpha, \chi^A] + \partial_\mu\tilde{\Omega}_\mu[A_\mu^{[\alpha\beta]};\chi^a] \\
\tilde{\mathcal{V}} &= \mathcal{V} + \mathcal{C}_d \\
\tilde{\Omega} &= \Omega - \Omega_{CS} \ .
\end{aligned} \tag{21}$$

Clearly, $\tilde{\mathcal{V}}$ is *gauge invariant*. This enables the definitions of the topological charge densities, for odd and even d respectively, as

$$\begin{aligned}
\varrho &= \varrho_G - \mathcal{V} && (22) \qquad & \varrho &= \varrho_G - \tilde{\mathcal{V}} && (24) \\
&= \varrho_0 + \nabla.\mathbf{\Omega} && (23) \qquad & &= \varrho_0 + \nabla.\tilde{\mathbf{\Omega}} \ . && (25)
\end{aligned}$$

It follows from (23) and (25) that the volume integral of ϱ equals the *widing number*, or, the *degree of the map*. At the same time it follows from (22) and (24) that ϱ is gauge invariant, so that the latter definitions of the topological charge can be employed to establish Bogomol'nyi inequalities supplying topological lower bounds on the actions/energy of candidate models.

The d dimensional gauged $O(d+1)$ sigma models (supporting solitons) have certain features contrasting with the d dimensional Higgs and gauged Grassmannian models (supporting solitons) discussed above.

- For the Higgs and gauged Grassmannian models the soliton does not survive the gauge decoupling limit while for the gauged $O(d+1)$ models it does.
- For the Higgs models the gauge group must be $SO(d)$, while for the gauged $O(d+1)$ models it can be $SO(N)$ for any $2 \leq N \leq d$.
- For the Higgs models the finite action/energy conditions fix the asymptotic values of the gauge field functions *uniquely*, while for the gauged $O(d+1)$ models the gauge field functions in general turn out to be *multi-valued* asymptotically. As a result the solutions feature certain bifurcation patterns [20] which may be of some physical interest.

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