Quasi-localized states on noncommutative solitons

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Abstract

We consider noncommutative gauge theories which have zero mass states propagating along both commutative and noncommutative dimensions. Solitons in these theories generically carry $U(m)$ gauge group on their world-volume. We show that once the world-volume $U(m)$ gauge theory is in the Higgs phase, light states become quasi-localized, rather than strictly localized on the soliton, *i.e.* they mix with light bulk modes and have finite widths to escape into the noncommutative dimensions. At small values of $U(m)$ symmetry breaking parameters, these widths are small compared to the corresponding masses.

1 Introduction and summary

Quasi-localization, rather than perfect localization, of states on a brane is a common property of various brane-world models [1, 2, 3, 4, 5, 6]. Particles may not be trapped to a brane forever, but may have finite, albeit small, probability to escape into extra dimensions. This phenomenon may occur even at low energies, provided bulk modes have continuum spectrum starting from zero energy and there is mixing between brane modes and continuum modes (for a review see, e.g., Ref. [7]). Clearly, this possibility is of interest for phenomenology, and also for the study of the properties of branes in a more theoretical context.

Particularly interesting field theory branes are noncommutative solitons (for reviews see Refs. [8, 9, 10]). Noncommutative field theory arises in an appropriate limit of string theory [11], and the properties of many noncommutative solitons match nicely the properties of D-branes. Indeed, it has been suggested [12] to use the noncommutative solitons for constructing phenomenologically acceptable brane-world models.

In this paper we discuss quasi-localization of states on solitons in noncommutative gauge theories. The class of models we consider is the one where the gauge (and possibly matter) fields have continuum of bulk modes, weakly coupled at low energies[1]. From the point of view of string theory, solitons in these models correspond to "branes within branes", see, e.g., Ref. [15] and references therein.

Generically, a soliton of this type has $U(m)$ gauge symmetry on its world-volume. If (part of) this gauge symmetry is unbroken, charged matter fields as well as massless gauge fields are strictly localized on the soliton. Our main observation is that the situation changes if $U(m)$ on the soliton world-volume is in the Higgs phase: gauge and/or matter fields become quasi-localized. For fields that have massless bulk modes, escape into extra dimensions occurs even

[1] We live aside unstable solitons corresponding to D-branes in tachyonic vacuum [13, 14].

at low energies; at small value of the parameter of $U(m)$ symmetry breaking, the life-times against this escape are large compared to the inverse masses of the quasi-localized modes.

To introduce the mechanism of quasi-localization most explicitly, we consider in Sect.2 a simple example of a m-vortex solution in $U(1)$ gauge-Higgs theory in two noncommutative and p commutative spatial dimensions [16, 17, 18, 19]. The bulk modes of both gauge and Higgs fields are massive in this model, so escape of these fields into the noncommutative dimensions does not occur at low energies. We introduce an extra adjoint scalar field which has massless modes in the bulk, and show that its states become quasi-localized on the vortex even at low energies, provided that $U(m)$ gauge theory on the vortex is in the Higgs phase. We calculate the widths of the quasi-localized states at small values of the parameter governing $U(m)$ symmetry breaking, and find that these widths are parametrically smaller than the masses of these states. In this model there is a hierarchy of life-times of different quasi-localized modes. In the case of vortex, this hierarchy is related to the rotational symmetry of the background: we will see that higher angular momentum modes live *longer* on the soliton, because certain mixing terms of these modes are either forbidden by rotational symmetry or suppressed due to the centrifugal barrier.

In Sect.3 we study $U(2)$ pure gauge theory in four noncommutative and p commutative spatial dimensions. Instantons in this theory [20, 21, 22, 23, 24] correspond to $Dp-D(p+4)$ system. "Zero-size" anti-self-dual m-instantons in a theory with anti-self-dual noncommutativity [21, 23] (which are actually non-singular solutions) have unbroken, strictly localized $U(m)$ gauge theory on their world-volume. We show that once the instanton size is non-vanishing, the gauge theory on the instanton world-volume not only is in the Higgs phase, but also becomes quasi-localized even at low energies. We consider explicitly the case of small instanton size, which corresponds to small gauge boson masses on the soliton, and show that the widths of the quasi-localized gauge bosons against the escape into noncommutative dimensions are again small compared to the masses of these states.

It appears that the quasi-localization of low-energy theory on noncommutative solitons is generic in models having massless modes in the bulk and gauge theories in the Higgs phase on the soliton world-volume. It is tempting to speculate that in string theory motivated brane-world models, massive particles which are neutral under electric charge and color may be unstable against escape into extra dimensions. On the other hand, in the context of noncommutative theories without gravity, particles carrying unbroken charges of the brane-world gauge theory, and also massless gauge bosons of that theory are trapped to the solitonic brane forever (in other words, processes like $e^- \to$ *nothing* or $\gamma \to$ *nothing* are not allowed, unlike in some other brane-world models [25, 4, 6]). It remains to be understood whether or not this property still holds when gravity is turned on.

2 Quasi-localization on noncommutative vortex

2.1 Vortex solutions

In this section we consider an $U(1)$ gauge theory with fundamental Higgs field in $(1_{\rm time}+p+2)$-dimensional space-time with two space-like noncommutative dimensions $x_1,\ x_2$. The action for this theory has the following form,

$$S=\frac{1}{g^2}\int d^{p+1}y\, d^2x\left[-\frac{1}{4}F_{AB}*F^{AB}+D_A\phi^+*D^A\phi-\frac{1}{2}(\phi^+*\phi-v^2)^2\right] \tag{1}$$

where y^μ are commuting dimensions,

$$\begin{aligned} F_{AB} &= \partial_A A_B-\partial_B A_A-i(A_A*A_B-A_B*A_A) \\ D_A\phi &= \partial_A\phi-iA_A*\phi \end{aligned} \tag{2}$$

and the Moyal product is defined as follows,

$$f(x)*g(x)\equiv \mathrm{e}^{-i\frac{\theta}{2}\epsilon^{ij}\partial_i\partial'_j}f(x)g(x')\Big|_{x=x'} .$$

As shown in Ref. [16, 17, 18, 19], this theory admits static soliton solutions independent of the commuting coordinates and having a form of a vortex in the noncommutative plane x_1, x_2. To describe these solitons, let us switch to the Fock space notations. Then the energy density takes the following form,

$$E=\frac{2\pi\theta}{g^2}\mathrm{Tr}\left\{\frac{1}{2\theta^2}([C,C^+]+1)^2+D_z\phi^+D_{\bar z}\phi+D_{\bar z}\phi^+D_z\phi+\frac{1}{2}(\phi^+\phi-v^2)^2\right\} ,$$

where

$$z=\sqrt{\frac{1}{2}}(x_1+ix_2)\ ,\quad [z,\bar z]=\theta$$
$$C=a^++i\sqrt{\theta}A_z$$
$$a=\frac{z}{\sqrt{\theta}},\ a^+=\frac{\bar z}{\sqrt{\theta}}$$
$$D_z\phi=-\frac{1}{\sqrt{\theta}}[a^+,\phi]-iA_z\phi\ ,\qquad D_{\bar z}\phi=\frac{1}{\sqrt{\theta}}[a,\phi]-iA_{\bar z}\phi$$

The properties of stable vortex solutions depend on the value of the parameter θv^2. At $\theta v^2\geq 1$ stable vortex is non-BPS and can be obtained by the solution generation technique [18, 19, 23]. The m-soliton solution has the following form in this case (for all solitons located at one point in the noncommutative plane)

$$\phi=vS_m^+\ ,\qquad C^+=S_m^+aS_m \tag{3}$$

where S_m^+ is the shift operator,

$$S_m^+ = \sum_{n=0}^{\infty} |n+m\rangle\langle n| \ .$$

When considered as a p-brane in $(p+2)$-dimensional space, this soliton carries on its world-volume localized gauge fields corresponding to the unbroken $U(m)$ subgroup of the original $U(\infty)$ group. There is also continuous spectrum of gauge fields corresponding to the broken generators of $U(\infty)$. The latter fields propagate in the bulk. Non-zero vacuum expectation value of the field ϕ at $|z| \to \infty$ (see Eq. (3)) provides a mass gap gv to this continuum.

At $\theta v^2 < 1$ the solution given by Eq. (3) is unstable. Instead, there exists a BPS solution of the following form [19]

$$\phi = v \sum_{n=0}^{\infty} (1+\phi_n)|n+m\rangle\langle n| \ , \qquad C^+ = \mu_m + \epsilon_m + S_m^+ \tilde{a} S_m \ ,$$

where

$$\mu_m \equiv \sum_{\alpha=1}^{m-1} \sqrt{\alpha(1-\theta v^2)}|\alpha-1\rangle\langle\alpha| \ , \qquad \epsilon_m \equiv \sqrt{m(1-\theta v^2)}|m-1\rangle\langle m| \ , \tag{4}$$

$$\tilde{a} \equiv \sum_{n=1}^{\infty} (\sqrt{n} + c_n)|n-1\rangle\langle n| \ . \tag{5}$$

The corresponding matrices in the Fock basis are

$$C^+ = \left(\begin{array}{cccccccc} \multicolumn{5}{c}{\overbrace{\hphantom{0 \quad \sqrt{\omega} \quad 0 \quad \dots \quad 0}}^{m}} & & & \\ 0 & \sqrt{\omega} & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2\omega} & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{(m-1)\omega} & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & \sqrt{m\omega} & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & & & \\ \dots & \dots & \dots & \dots & \dots & & \tilde{a} & \\ 0 & 0 & 0 & \dots & 0 & & & \end{array}\right) \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} m \\ \\ \\ \\ \end{array} \tag{6}$$

$$\phi = \left(\begin{array}{cccc} 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots \\ v_1 & 0 & 0 & \dots \\ 0 & v_2 & 0 & \dots \\ 0 & 0 & v_3 & \dots \\ \dots & \dots & \dots & \dots \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} m \\ \\ \\ \\ \\ \end{array} \tag{7}$$

where

$$\omega \equiv \sqrt{1-\theta v^2}$$

and $v_n = v(1+\phi_n)$. The coefficients ϕ_n and c_n are determined by a set of recursion relations which were obtained in Ref. [19]. In what follows we consider the case when the parameter ω is small,

$$\omega \ll 1 \ .$$

In this case ϕ_n and c_n are also small and their explicit form is not essential for our purposes. The relevant property of these coefficients is that they tend to zero as n tends to infinity,

$$\lim_{n\to\infty} \phi_n, c_n = 0 \ . \tag{8}$$

μ_m and ϵ_m serve as vacuum expectation values of the adjoint and fundamental Higgs fields giving masses to the gauge bosons of the $U(m)$ gauge group on the vortex. As a result, this gauge group is spontaneously broken completely. In addition, ϵ_m introduces mixing between $U(m)$ gauge bosons and gauge bosons from the continuum spectrum. However, the latter has a mass gap, the gauge bosons from continuum can be integrated out at low energies, and this mixing does not lead to any interesting low energy effects at $\omega \ll 1$.

2.2 Adjoint scalar

Let us now introduce additional massless real adjoint scalar field f with the action

$$\frac{2\pi\theta}{g^2}\int d^{p+1}y \ \mathrm{Tr}\left\{-\frac{1}{\theta}[C,f][C^+,f] + D_\mu f (D_\mu f)^+\right\} \ , \tag{9}$$

where

$$D_\mu f = \partial_\mu f - i[A_\mu, f] \ .$$

Let us study the mass spectrum of the field f in the vortex background. A non-trivial mass matrix for f is provided by the first term in the action (9). It is convenient to decompose the field f in the following way,

$$f = \left(\begin{array}{cc} \overbrace{\psi}^{m} & \xi \\ \xi^+ & \chi \end{array}\right) \ \}m \tag{10}$$

In other words,

$$f = \psi + \xi + \xi^+ + \chi \ ,$$

where ψ is a Hermitean $m\times m$ matrix

$$\psi = P_m f P_m \equiv \sum_{\alpha,\beta=0}^{m-1} \psi_\alpha^\beta |\alpha\rangle\langle\beta| \ ,$$

ξ is a $m \times \infty$ matrix

$$\xi = P_m f(1-P_m) \equiv \sum_{\alpha=0}^{m-1} \sum_{n=0}^{\infty} \xi_\alpha^n |\alpha\rangle\langle m+n| \ ,$$

and χ is a Hermitean $\infty \times \infty$ matrix

$$\chi = (1-P_m)f(1-P_m) \equiv \sum_{k,n=0}^{\infty} \chi_k^n |m+k\rangle\langle m+n| \ .$$

Here P_m is a projector

$$P_m = \sum_{\alpha=0}^{m-1} |\alpha\rangle\langle\alpha| \ .$$

In the case of non-BPS vortex (3) with unbroken $U(m)$ gauge group on its world-volume, it is straightforward to check that the fields $\psi_\alpha^\beta(y)$ are massless scalar fields strictly localized on the vortex world-volume and belonging to adjoint representation of $U(m)$, $\xi_\alpha^n(y)$ are massive fundamentals with masses

$$m_n^2 = \frac{2n+1}{\theta} \tag{11}$$

and $\chi_k^n(y)$ are fields in the gapless continuum spectrum, which are neutral under $U(m)$.

2.3 Quasi-localization on single vortex

Let us first discuss the spectrum of the adjoint scalar field at small but non-zero ω in the background of one-vortex solution, $m = 1$. In the matrix notations (6) and (10), the first term in the action (9) takes the following form,

$$\begin{aligned} -\frac{1}{\theta}\mathrm{Tr}\left\{[C,f][C^+,f]\right\} = &-\frac{1}{\theta}\Big[\mathrm{Tr}\left\{[\tilde{a},\chi][\tilde{a}^+,\chi]\right\} \\ &+\sum_{n=1}^{\infty}(2\tilde{n}_1-1)\xi^{n-1}\xi^*_{n-1} + \sum_{n=1}^{\infty}\omega^2\chi_0^n\chi_n^0 + \omega^2\chi_0^0\chi_0^0 + \omega^2\psi^2 \\ &-\sum_{n=1}^{\infty}2\omega\sqrt{\tilde{n}_2}(\chi_n^0\xi^{n-1}+\chi_0^n\xi^*_{n-1}) - 2\omega^2\chi_0^0\psi\Big] \end{aligned} \tag{12}$$

where

$$\tilde{n}_{1,2} = n + O(\omega) \ .$$

The first term in Eq. (12) describes the propagation of the field χ along noncommutative dimensions: with appropriate redefinition of noncommutative coordinates it becomes merely the gradient term, $\int d^2z' \partial_{z'}\chi\partial_{\bar{z}'}\chi + O(\omega)$. Due to the property (8), the propagation is free far away from the vortex, and the spectrum still starts from zero energy. The second term in Eq. (12) gives large masses (11) (up to small corrections) to the fundamentals ξ^n. The third, fourth and fifth terms provide small diagonal masses to the fields χ_0^n, χ_0^0 and ψ, respectively. Most importantly, there is a term that mixes the would-be localized state ψ with the states from continuum: this is the last term in Eq. (12). Integrating out massive fields ξ^n one arrives at the following effective mass terms for light fields χ_n^k and ψ (neglecting $O(\omega)$ corrections in $\tilde{n}_{1,2}$)

$$\begin{aligned} M = &-\frac{1}{\theta}\mathrm{Tr}\left\{[\tilde{a},\chi][\tilde{a}^+,\chi]\right\} + \sum_{n=0}^{\infty}\frac{\omega^2}{\theta}\left(1-\frac{4n}{2n-1}\right)\chi_0^{n*}\chi_0^n \\ &- 2\frac{\omega^2}{\theta}\chi_0^0\psi + \frac{\omega^2}{\theta}\psi^2 \ . \end{aligned} \tag{13}$$

This effective mass terms are relevant at energy scales below $1/\sqrt{\theta}$. One observes that at small ω, there is an interesting low energy scale $\omega/\sqrt{\theta}$, so we concentrate on physics at this scale.

We begin with the first two terms in Eq. (13). They contain corrections to the quadratic action of the fields χ_0^n, which at first glance appear relevant at the scale $\omega/\sqrt{\theta}$. Let us see that this is not the case.

Let us come back to coordinate formulation of the noncommutative theory, and write the field $\chi(y,z,\bar{z})$ in Fourier representation along noncommutative dimensions,

$$\chi(y,z,\bar{z}) = \int\frac{d^2k}{(2\pi)^2}\tilde{\chi}_k(y)e^{i(k_z z + k_{\bar{z}}\bar{z})} \ . \tag{14}$$

We are interested in low-momentum part, $k \sim \omega/\sqrt{\theta}$. The components χ_0^n entering the second term in Eq. (13) are

$$\chi_0^n = \frac{i^n}{\sqrt{n!}}\int\frac{d^2k}{(2\pi)^2}\mathrm{e}^{-k^2\theta/2}\tilde{\chi}_k(y)(k_{\bar{z}}\sqrt{\theta})^n \ \approx \ \frac{i^n}{\sqrt{n!}}\int\frac{d^2k}{(2\pi)^2}\tilde{\chi}_k(y)(k_{\bar{z}}\sqrt{\theta})^n \tag{15}$$

Since $d^2k \propto \omega^2/\theta$, the second term in Eq. (13) is at least of order $(\omega^6/\theta^3\cdot|\tilde{\chi}_k|^2)$, whereas the gradient term is of order

$$\frac{1}{\theta}\int d^2k\ k^2|\tilde{\chi}_k|^2 \sim \omega^4/\theta^3\cdot|\tilde{\chi}_k|^2 \ .$$

Thus, the second term in Eq. (13) is small at small ω and $k \sim \omega/\sqrt{\theta}$. Similar argument applies to the corrections to the gradient term coming from the fact that $\tilde{a}$ and a differ by $O(\omega^2)$.

Neglecting the higher-order terms, we write the low energy effective action as follows,

$$\begin{aligned} S_{eff} = \frac{2\pi\theta}{g^2}\int d^{p+1}y\,\Bigg(&(\partial_\mu\psi)^2 - \frac{\omega^2}{\theta}\psi^2 + \\ &+\int\frac{d^2k}{2\pi\theta(2\pi)^2}\left(\partial_\mu\tilde{\chi}_k\partial_\mu\tilde{\chi}_{-k} - k^2\tilde{\chi}_k\tilde{\chi}_{-k} + 4\pi\omega^2\tilde{\chi}_k\psi\right)\Bigg) \ . \end{aligned} \tag{16}$$

From this action one obtains the following equations determining the propagator $G_\psi(p)$ of the field ψ,

$$\left(p^2 - \frac{\omega^2}{\theta}\right) G_\psi(p) + \frac{\omega^2}{\theta}\int \frac{d^2k}{(2\pi)^2}\tilde{\chi}_k \quad = \quad 1 \tag{17}$$

$$\left(p^2 - k^2\right)\tilde{\chi}_k + 2\pi\omega^2 G_\psi(p) \quad = \quad 0 \tag{18}$$

where p^μ is the momentum along commutative dimensions. Expressing $\tilde{\chi}_k$ from Eq. (18) and substituting it into Eq. (17) one obtains that the propagator $G_\psi(p)$ has the Breit–Wigner form

$$G_\psi(p) = \frac{1}{p^2 - m^2 + im\Gamma}$$

with mass

$$m^2 = \frac{\omega^2}{\theta}$$

and width

$$\Gamma = -\frac{\omega^3}{2\pi\sqrt{\theta}}\mathrm{Im}\int \frac{d^2k}{p^2 - k^2 + i\epsilon} = \frac{\pi\omega^3}{2\sqrt{\theta}} \ll m \tag{19}$$

Thus, mixing between the field ψ and fields $\tilde{\chi}_k$ from the continuum spectrum results in the delocalization of the field ψ. This field no longer describes a stable state localized on the vortex. Rather it corresponds to a metastable resonance embedded in the continuum spectrum. This state has a small but non-vanishing probability to escape from the brane.

2.4 Multi-vortex case: hierarchy of widths

Let us now consider the multi-vortex solution with $m > 1$. We still study the case $\omega \ll 1$, and physics at energy scale $\omega/\sqrt{\theta}$. To understand what happens with field f in this case, it is convenient to make use of the symmetry under rotations in the noncommutative plane, which is present in the action (9) when the background field C^+ is given by Eq. (4). Namely, this action is invariant under the transformations

$$f \to \mathrm{e}^{-i\alpha a^+ a} f \mathrm{e}^{i\alpha a^+ a} ,$$

leaving field C invariant. Explicitly, this rotation acts on the matrix elements of the operator f as follows,

$$f^n_m \to \mathrm{e}^{i\alpha(n-m)} f^n_m$$

where f^n_m are defined by

$$f = \sum_{m,n=0}^{\infty} f^n_m |m\rangle\langle n| \,.$$

In other words, the field $f^n_m(y)$ has charge $(n-m)$ under this symmetry. Clearly, this charge can be interpreted as the angular momentum in the noncommutative plane. Consequently, the action (9) in the background field (4) is the sum of the actions for fields with different angular momenta. The fields with angular momentum l combine into the matrix

$$f=\begin{pmatrix} \overbrace{0 \ \dots \ 0 \ f_0^l}^{l} & 0 & \dots\dots\dots\dots & & \\ 0 \ \dots\dots \ 0 & f_1^{(l+1)} & 0 & \dots\dots & \\ 0 \ \dots\dots\dots & 0 & f_2^{(l+2)} & 0 & \dots \\ \dots\dots\dots\dots\dots\dots & & & & \end{pmatrix} + h.c. \tag{20}$$

Due to the rotational symmetry, the fields with different l decouple. Furthermore mixing between the states occurs between neighboring entries of this matrix.

In terms of the fields ψ, ξ and χ introduced in Eq. (10), the latter property implies that the would-be bound states ψ with non-zero angular momentum do not mix directly to the continuum states χ. Indeed, non-trivial mixing occurs between the neighboring entries of the matrix (omitting indices of the fields),

$$\begin{array}{l} m-l\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ l\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \\ \\ \end{array} \begin{pmatrix} \overbrace{0 \ \dots \ 0 \ \psi}^{l} \ \overbrace{0 \ \dots}^{m-l} & \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \\ 0 \ \dots\dots \ 0 \ \psi & 0 \ \dots\dots\dots\dots \\ 0 \ \dots\dots\dots \ 0 & \xi \ 0 \ \dots\dots\dots \\ \dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \\ 0 \ \dots\dots\dots\dots & \dots \ 0 \ \xi \ 0 \ \dots\dots \\ 0 \ \dots\dots\dots\dots & \dots\dots \ 0 \ \chi \ 0 \ \dots \\ 0 \ \dots\dots\dots\dots & \dots\dots\dots \ 0 \ \chi \ 0 \ \dots \\ \dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \end{pmatrix} + h.c. \tag{21}$$

Thus, the fields ψ mix between themselves and with heavy fields ξ only, and the decay of ψ into continuum states occurs only through weak mixing of the heavy fields ξ between themselves and finally with the fields χ propagating along noncommutative dimensions. Clearly, this introduces the suppression of the decay widths of the quasi-localized states ψ with angular momentum l by extra factor ω^{2l}. Yet another suppression occurs due to the fact that at low momenta, the components χ_m^n with $(n-m)=l\neq 0$ are small, essentially due to the centrifugal barrier.

Proceeding similar to the case of a single vortex one obtains the following estimate for the widths of the components of the field ψ of angular momentum l,

$$\Gamma \propto \frac{\omega^{4l+3}}{\sqrt{\theta}} \,. \tag{22}$$

We conclude that at small θ, all quasi-localized states have masses of order $\omega/\sqrt{\theta}$, but there is a hierarchy between their widths: modes with higher angular momenta live longer on the soliton.

3 Quasi-localization on noncommutative instantons

In this section we study quasi-localization of massive gauge fields on noncommutative instantons. We begin with one-instanton case in $U(2)$ noncommutative gauge theory and make use of the explicit solution found in Ref. [24]. We then generalize to $U(2k)$ noncommutative gauge theory and consider simple k-instantons. These support $U(k)$ gauge theory on their world-volume. Once the k-instanton background is such that this $U(k)$ gauge theory is in the Higgs phase, the massive gauge bosons become quasi-localized.

3.1 One-instanton solution

Let us describe the one-instanton solution in the $U(2)$ noncommutative pure gauge theory, which was explicitly constructed in Ref. [24]. One considers the $U(2)$ gauge theory in $(1_{time} + p + 4)$-dimensional space-time with four space-like noncommutative dimensions $z, \bar{z}, \zeta, \bar{\zeta}$ and commutative dimensions y^μ. Following Ref. [24] we consider the case of anti-self-dual parameter of noncommutativity, *i.e.*,

$$[z, \bar{z}] = -[\zeta, \bar{\zeta}] \equiv \theta > 0 \ .$$

In the Fock basis, the action for this theory has the following form,

$$S = (2\pi\theta)^2 \int d^{p+1}y \ \mathrm{Tr} \left[-\frac{1}{4} F_{ij}F^{ij} + \frac{1}{2\theta} D_\mu C_{\bar{\zeta}} D^\mu C_\zeta + \frac{1}{2\theta} D_\mu C_{\bar{z}} D^\mu C_z - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right] \quad (23)$$

where $i = z, \bar{\zeta}$, C_i are 2×2 matrices whose entries are operators acting in the Fock space of two-particle quantum mechanics. Components of the field strength F_{ij} along noncommutative dimensions are determined by these matrices in the usual way

$$F_{z\bar{z}} = -\frac{1}{\theta}([C_z, C_z^+] + 1) \ , \qquad F_{\zeta\bar{\zeta}} = \frac{1}{\theta}([C_{\bar{\zeta}}, C_{\bar{\zeta}}^+] + 1)$$
$$F_{\zeta z} = -\frac{1}{\theta}[C_{\bar{\zeta}}^+, C_z] \ , \qquad F_{\zeta\bar{z}} = -\frac{1}{\theta}[C_{\bar{\zeta}}^+, C_z^+] \ .$$

The covariant derivative of the field C_z is

$$D_\mu C_z = \partial_\mu C_z - i[A_\mu, C_z] \quad (24)$$

and the same for the field $C_{\bar{\zeta}}$. Recall that $C_{\bar{z}} = C_z^+$, $C_\zeta = C_{\bar{\zeta}}^+$, and that the vacuum is $C_z = a_z^+ \cdot \mathbf{1}$, $C_{\bar{\zeta}} = a_\zeta^+ \cdot \mathbf{1}$, where $\mathbf{1}$ is 2×2 unit matrix (which we will not write explicitly in what follows).

The instanton solution is independent of commutative coordinates, has μ-components of the gauge field equal to zero and has anti-self-dual field strength. As in the commutative case, the Pontryagin index of the m-instanton solution,

$$N_P = -\frac{\theta^2}{8} \epsilon^{ijkl} \mathrm{Tr} F_{ij} F_{kl} \ , \quad (25)$$

is equal to m. In Eq. (25) the trace is evaluated over both the $U(2)$ indices and the Fock space.

A powerful tool for describing the moduli space of m-instanton solutions and obtaining explicit formulas for instanton fields is the noncommutative version [20] of the ADHM construction [26]. Explicit construction of the one-instanton solution in the noncommutative $U(2)$ theory involves the 4×2 matrix Ψ [24] which can be written in the following form,

$$\Psi = \left(\begin{array}{c} S^{+} \frac{\rho}{\sqrt{N+1+\rho^2}} S + 1 - S^{+} S \\ \\ \frac{\sqrt{N+1}}{\sqrt{N+1+\rho^2}} S \end{array} \right) \tag{26}$$

where entries are 2×2 matrices, the matrix U is

$$S = \frac{1}{\sqrt{N+1}} \left(\begin{array}{cc} a_{\zeta}^{+} & a_z \\ -a_z^{+} & a_{\zeta} \end{array} \right) , \tag{27}$$

and N is the occupation number operator, $N = a_z^{+} a_z + a_{\zeta}^{+} a_{\zeta}$. The real parameter ρ is natural to interpret as the size of the instanton in units of $\sqrt{\theta}$. The gauge fields $C_z(\rho)$, $C_{\bar{\zeta}}(\rho)$ of the instanton of size ρ are

$$C_z(\rho) = \Psi^{+} a_z^{+} \Psi \ , \qquad C_{\bar{\zeta}}(\rho) = \Psi^{+} a_{\zeta}^{+} \Psi \ .$$

It is straightforward to check that the field strength obtained from $C_z(\rho)$, $C_{\bar{\zeta}}(\rho)$ is anti-self-dual and has unit Pontryagin index [24].

The operator S is a partial isometry operator, *i.e.*

$$SS^{+} = 1 \ , \qquad S^{+} S = 1 - P_0 \ ,$$

where P_0 is the projector on the state

$$|\alpha\rangle = \left(\begin{array}{c} 0 \\ |0\rangle \end{array} \right) . \tag{28}$$

The instanton of zero size, $\rho = 0$, is non-singular and may be obtained from vacuum by the solution generation technique,

$$C_z(\rho = 0) = S^{+} a_z^{+} S \ , \qquad C_{\bar{\zeta}}(\rho = 0) = S^{+} a_{\zeta}^{+} S$$

Consequently, the instanton of zero size supports unbroken $U(1)$ gauge group on its world-volume.

3.2 Quasi-localization on single instanton

Let us study the spectrum of the components A_μ along the commutative directions, in the background of the instanton of small but non-vanishing size ρ.

The operator ψ which describes the would-be zero mode of A_μ is (in what follows we drop the index μ everywhere)

$$\psi = \psi_0(y) \begin{pmatrix} 0 & 0 \\ 0 & |0\rangle\langle 0| \end{pmatrix} . \tag{29}$$

The effective action which governs its dynamics at low energies contains (cf. Eq. (13)) the mass term:

$$M_{\psi\psi} = \frac{2\rho^2}{\theta(1+\rho^2)}\psi_0^2 \tag{30}$$

and direct mixing between the field $\psi_0(y)$ and continuum

$$M_{\psi\chi} = -\frac{2\rho^2}{\theta(1+\rho^2)}\psi_0 \left(\langle\alpha|\chi|\alpha\rangle + \langle u|\chi|u\rangle\right) , \tag{31}$$

where

$$|u\rangle = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} . \tag{32}$$

There is no first order (in ρ expansion) mixing between the light mode ψ and heavy modes, thus mixing involving the heavy states is negligible at small ρ and low energies.

The rest of the analysis is the same as in the case of vortex. The effect of the two terms (30), (31) is that the field ψ_0 describes a quasi-localized massive vector field on the soliton world-volume, whose mass and width (at small ρ) are

$$m_0^2 = \frac{2\rho^2}{\theta} , \qquad \Gamma = \frac{\pi\sqrt{2}\rho^5}{\sqrt{\theta}} , \tag{33}$$

The extra factor ρ^2 in the expression for the width, as compared to the case of vortex (cf. Eq. (19), is due to the fact that there are four, rather than two, transverse dimensions in the case of instanton.

3.3 Multi-instanton case

In this subsection we discuss quasi-localization on a multi-instanton. Similar to the case of vortex, the main difference from the one-instanton case is that an m-instanton solution supports a *non-Abelian* gauge group $U(m)$ on its world-volume. If all m instantons sit on top of each other, this $U(m)$ is unbroken. One way to break this gauge group spontaneously is to move to a general point in the Coulomb branch where positions of instantons in the noncommutative hyperplane are not coincident. This splitting leaves unbroken a subgroup $[U(1)]^m$

of $U(m)$. The massive gauge bosons on the soliton all carry non-zero charges corresponding to some of the $U(1)$ factors, so they remain strictly localized.

We consider instead the case in which instantons have non-zero sizes. This corresponds to the Higgs branch of the instanton moduli space. A simple solution of this kind may be obtained in $U(2k)$ gauge theory by making use of the one-instanton solution considered in the previous subsection. For the sake of simplicity, let us consider two-instanton solution in the $U(4)$ gauge theory; a generalization to k-instanton solution in $U(2k)$ gauge theory is straightforward. The gauge field of a simple anti-self-dual solution describing two instantons of sizes ρ_1 and ρ_2 sitting on top of each other has the following block-diagonal form,

$$C_z(\rho_1, \rho_2) = \begin{pmatrix} C_z(\rho_1) & 0 \\ 0 & C_z(\rho_2) \end{pmatrix} , \tag{34}$$

and analogously for $C_{\bar{\zeta}}(\rho_1, \rho_2)$. Here $C_z(\rho_i)$ are 2×2 matrices describing one-instanton solution in the $U(2)$ gauge theory, see Eq. (3.1). Clearly, the field strength corresponding to Eq. (34) is anti-self-dual and has Pontryagin index equal to two.

When both instantons have zero sizes, $\rho_1 = \rho_2 = 0$, this solution may be obtained from vacuum by the solution generation technique with the partial isometry operator

$$S_2 = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} , \tag{35}$$

where S is given by Eq. (27). In this case there is an unbroken $U(2)$ gauge group, which corresponds to unitary transformations in the two-dimensional subspace V_0 of the Fock space, whose basis vectors are

$$|\alpha_1\rangle = \begin{pmatrix} |\alpha\rangle \\ 0 \end{pmatrix} , \quad |\alpha_2\rangle = \begin{pmatrix} 0 \\ |\alpha\rangle \end{pmatrix} , \tag{36}$$

where the two-column $|\alpha\rangle$ is given by Eq. (28). The four real zero modes ψ of the field A_μ, corresponding to this gauge group, can be organized as follows,

$$\psi = \begin{pmatrix} \psi_1^1(y) & \psi_1^2(y) \\ \psi_2^1(y) & \psi_2^2(y) \end{pmatrix} \otimes |\alpha\rangle\langle\alpha| , \tag{37}$$

where $\psi_1^2 = \psi_2^{1*}$, and ψ_1^1, ψ_2^2 are real.

When both instantons have small but non-zero sizes, $\rho_1 \neq 0$, $\rho_2 \neq 0$, the $U(2)$ gauge group is completely Higgsed, and all its gauge fields become massive. Their mass matrix is obtained by plugging the fields (37) and (34) into the action with the result

$$M_{\psi\psi} = \frac{2}{\theta} \left(\frac{\rho_1^2}{1+\rho_1^2} (\psi_1^1)^2 + \frac{\rho_2^2}{1+\rho_2^2} (\psi_2^2)^2 + \left(\frac{\rho_1^2}{1+\rho_1^2} + \frac{\rho_2^2}{1+\rho_2^2} \right) \psi_1^2 \psi_2^1 \right) \tag{38}$$

Similarly to the one-instanton case, there is no mixing between the fields ψ_α^β and heavy charged fields $|\xi_\alpha\rangle$ to the linear order and the leading contributions to the widths of ψ_α^β come from direct mixing with the fields from the continuum. The diagonal components ψ_1^1 and

ψ_2^2 mix with the corresponding diagonal components χ_1^1 and χ_2^2. Each of these mixings has precisely the same form as in the one-instanton case, Eq. (31), leading to the widths

$$\Gamma_{11} = \frac{\pi\sqrt{2}\rho_1^5}{\sqrt{\theta}}, \quad \Gamma_{22} = \frac{\pi\sqrt{2}\rho_2^5}{\sqrt{\theta}} \,. \qquad (39)$$

The off-diagonal component ψ_1^2 mixes with the off-diagonal component χ_2^1 of the field χ. This mixing has the following form

$$M_{\psi\chi}^{off-diag} = -\frac{2\rho_1\rho_2}{\theta\sqrt{(1+\rho_1^2)(1+\rho_2^2)}}\psi_1^2\left(\langle\alpha|\chi_2^1|\alpha\rangle + \langle u|\chi_2^1|u\rangle\right) + h.c. \,, \qquad (40)$$

where $|u\rangle$ is still given by Eq. (32). This mixing leads to the width of ψ_1^2,

$$\Gamma_{12} = \frac{\pi\sqrt{2}(\rho_1\rho_2)^2(\rho_1^2+\rho_2^2)^{1/2}}{\sqrt{\theta}}. \qquad (41)$$

We see that when one of the instantons has zero size, one of the diagonal gauge bosons is massless, which is related to the fact that the $U(2)$ gauge group on the instanton world-volume is broken down to its $U(1)$ subgroup. In this case the off-diagonal component of the vector field is massive, see Eq. (38). However, its width is equal to zero. This component describes massive vector field charged under the unbroken $U(1)$ gauge group in this case. Thus vanishing of its width confirms our general statement that charged fields do not decay into the bulk.

Acknowledgments

The authors are indebted to F.L. Bezrukov for useful discussions. This work has been supported in part by RFBR grant 99-02-18410, CPG and SSLSS grant 00-1596626, CRDF grant (award RP1-2103), Swiss Science Foundation grant 7SUPJ062239. The work of S.S. was supported in part also under RFBR grant 01-02-06034.

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