Nonequilibrium evolution in scalar $O(N)$ models with spontaneous symmetry breaking

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1 Introduction

In quantum field theory the interaction is specified by a field potential $U(\phi)$. Perturbation theory is based on a "perturbative vacuum" which in lowest order is defined by the minimum, or one of the minima, of the tree level potential and on Green functions of quantum fluctuations around this minimum. If there are several minima we have several sectors of the theory which do not communicate within the perturbative framework. The form of the potential outside the neighbourhood of the minima is not relevant for the successful prediction of scattering cross sections and other experimentally accessible quantities. More generally, once one considers thermal or quantum corrections, the basic properties of the system (including, e.g., phase transitions) are determined by the effective potential, again the neighbourhoods of the minima are relevant for perturbation theory.

Once one considers quantum field theory beyond the perturbation theory, new phenomena emerge. A "false" vacuum based on a local, but not global, minimum may decay into the true vacuum via bubble nucleation at finite temperature, or by quantum tunneling via the bounce solutions; different gauge sectors of a theory may communicate via instantons. Here the shape of the potential is relevant for determining the classical solutions, while the effective potential, which includes quantum or thermal corrections, is replaced with the effective action. Another variety of phenomena emerges if the quantum system has an explicit time dependence; such situations arise in cosmology [1]. There a classical potential, the inflaton potential is assumed to determine the inflationary stage of expansion; later the time dependence of

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the temperature leads to a time dependence of the effective potential. While bubbles, bounces and instantons are local phenomena, here even a spatially homogeneous field can move with time and, e.g., may connect different sectors of the theory. Here the shape of the potential in a wide range of the mean field ϕ becomes relevant.

A type of potentials that has been wideley considered, are potentials with spontaneous symmetry breaking [2]. In particular they are relevant for the Higgs phenomenon, which is at the basis of the electroweak standard model, and which may be relevant to its GUT extensions. Prototypes are the double well and the Mexican hat potentials. Such potentials have regions of negative curvature where the squared mass of the physical Higgs boson becomes negative, and likewise regions where the squared mass of the Goldstone bosons becomes negative. The negative squared masses of the fundamental excitations signal an instability of the theory. So one cannot sustain a spatially homogeneous classical or mean field in regions where some mass becomes imaginary, instability implies the field becoming time-dependent. Therefore, if one wants to explore the meaning of the potential in the unstable regions one has to leave the realm of equilibrium quantum field theory and one is lead to consider quantum field theory out of equilibrium. This has been done by several groups, mainly within the last decade.

The theoretical framework for the real-time simulations of nonequilibrium quantum field theory is the Schwinger-Keldysh formalism [3], also referred to as closed-time-path (CTP) or real time formalism. The second ingredient for simulations is some approximation to the underlying quantum field theory. Approximations which include quantum back-reaction via resummation have been based on the on the Cornwall-Jackiw-Tomboulis (CJT or 2PI) formalism [4], as adapted to nonequilibrium quantum field theory by Calzetta and Hu [5]. Alternatively one may use the 2PPI formalism of Verschelde and Coppens [6]. Both formalisms are Φ-derivable and can be renormalized consistently. Within the CTP formalism it is mainly the large-N limit that has been investigated [7, 8, 9, 10], higher approximations as next-to-leading order large N (NLOLN)[11] or the bare vertex approximation [12] have been considered recently. The 2PPI formalism contains in leading order of a loop expansion the Hartree approximation as presented here [13]. The two-loop extension of this expansion has been studied recently [14].

In the following we will mainly refer to a picture of an $O(N)$ sigma model, where instead of the Higgs particle we have the sigma field, and with the pions as the Goldstone bosons. We will at first define the model and the nonequilibrium equations of motion in the Hartree approximation, then we will present the numerical results of such simulations and finally draw some conclusions. A more extensive presentation of our work is found in Ref. [13].

2 The $O(N)$ vector model

The $O(N)$ vector model with spontaneous symmetry breaking is defined by the Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\Phi}\,\partial^\mu \vec{\Phi} - \frac{\lambda}{4}\left(\vec{\Phi}^2 - v^2\right)^2 . \tag{1}$$

We consider a quantum system out of equilibrium that is characterized by a spatially homogenous background field. The fields are separated as

$$\Phi_a = \phi_a(t) + \eta_a(\mathbf{x}, t) \tag{2}$$

into a classical part $\phi_a = \langle \Phi_a \rangle$ and a fluctuation part η_a with $\langle \eta_a \rangle = 0$. Furthermore, in view of spatial translation invariance it is convenient to decompose the quantum fluctuations via

$$\eta_i(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_{0i}} \left[a_{\mathbf{k}} f_i(k,t) e^{i\mathbf{kx}} + a^\dagger_{\mathbf{k}} f_i^*(k,t) e^{-i\mathbf{kx}} \right] , \tag{3}$$

where $\omega_{0i} = \sqrt{k^2 + m_{0i}^2}$. m_{0i} will be defined below. The subscript $i = 1$ denotes the sigma mode ($a = 1$), $i = 2$ denotes the pion modes ($a = 2 \ldots N$).

In formulating the equations of motion and the renormalization we follow the presentation of Nemoto et al. [15] whose generalization to nonequilibriums system [13] is straightforward.

We introduce the inverse propagator in the classical background field in an $O(N)$ symmetric form

$$\mathcal{G}^{-1}_{ab} = \left[\Box + \mathcal{M}_2^2\right]\delta_{ab} + \frac{\phi_a \phi_b}{\vec{\phi}^2}\left[\mathcal{M}_1^2(t) - \mathcal{M}_2^2(t)\right] . \tag{4}$$

Here $\mathcal{M}_{1,2}$ are trial masses that will be determined self-consistently. In contrast to equilibrium quantum field theory these masses, as well as the classical field, are allowed to depend on time. The physical propagator will of course not be of this form. Indeed if one uses the CJT formalism in higher orders in the loop or the $1/N$ expansions, the propagator will have to be computed using a Schwinger-Dyson equation. The simple form of the propagator is preserved in higher orders in another resummation scheme, the 2PPI formalism [6].

If the inverse propagator has the restricted form characterized by just variational masses, the propagators themselves can be written in factorized form in terms of the mode functions $f_i(k,t)$. Then the Fock space defined by these mode functions and the associated Bogoliubov coefficients can be used to formulate a particle interpretation that has been widely used in cosmology or for particle creation in external fields. However, one has to be aware of the fact that these "particle excitations" are artefacts of the variational approximation and that their relation to physical particles has to be considered with care. So instead of particle numbers one should better consider the energy-momentum tensor or currents whose formulation does not depend on the particle picture. Keeping this in mind we may, superficially, associate $\mathcal{M}_1$ with the sigma mass and $\mathcal{M}_2$ with the pion mass. The fluctuations parallel to the vector $\vec{\phi}$ are the sigma fluctuations, the orthogonal ones are the pions.

In our application we choose the classical field to have just one nonvanishing component $\phi_1 \equiv \phi$; then the inverse propagator has only diagonal elements and these read

$$\mathcal{G}_{ii}(x,x') = \int \frac{d^3k}{(2\pi)^3 2\omega_{0i}} \exp\left(i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right) f_i(k,t_>) f_i^*(k,t_<) \tag{5}$$

(no summation over i).

In the Hartree approximation the equations of motion for the classical field and for the quantum fluctuations are given by

$$\begin{aligned} \ddot{\phi}(t) + \mathcal{M}_{\rm cl}^2(t)\phi(t) &= 0 \,, && (6) \\ \ddot{f}_i(k,t) + \left[k^2 + \mathcal{M}_i^2(t)\right] f_i(k,t) &= 0 \,. && (7) \end{aligned}$$

The masses $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy, for all t, the gap equations

$$\begin{aligned} \mathcal{M}_1^2(t) &= \lambda\left[3\phi^2(t) - v^2 + 3\mathcal{F}_1(t) + (N-1)\mathcal{F}_2(t)\right] && (8) \\ \mathcal{M}_2^2(t) &= \lambda\left[\phi^2(t) - v^2 + \mathcal{F}_1(t) + (N+1)\mathcal{F}_2(t)\right] && (9) \end{aligned}$$

and the "mass of the classical field" is given by

$$\mathcal{M}_{\rm cl}^2(t) = \mathcal{M}_1^2(t) - 2\lambda\phi^2(t) \,. \tag{10}$$

The quantities $\mathcal{F}_i(t)$ are the fluctuation integrals

$$\mathcal{F}_i(t) = \int \frac{d^3k}{(2\pi)^3 2\omega_{0i}} \left| f_i(k,t) \right|^2 . \tag{11}$$

The gap equations incorporate the resummation of bubble diagrams. The equations of motion and the gap equations contain divergent integrals and need to be replaced by renormalized ones. We have used the approach of Ref. [16], and we will assume that we work with the renormalized equations.

For a time-dependent problem we have to specify initial conditions. We choose at $t = 0$ a value of the classical field $\phi_0 = \phi(0)$ different from its value in the equilibrium ground state, which is given by v apart from quantum corrections. The initial mass parameters $m_{i0} = \mathcal{M}_i(0)$ are obtained by solving the gap equations (8) and (9) at $t = 0$, i.e., by finding the extremum of the effective potential at a fixed value of $\phi = \phi_0$. So the initial configuration is an equilibrium configuration with an externally fixed field ϕ_0. When the field is allowed, for $t > 0$, to become an internal dynamical field the nonequilibrium evolution sets in.

3 Numerical simulations

We will show here results for the "typical" case of $N = 4$ with a coupling $\lambda = 1$. We start the classical field at values $\phi_0 > v$, as for $\phi_0 < v$ the gap equations have no real solution. The region $\phi < v$ is only explored dynamically.

For the tree level potential the value $\phi = \sqrt{2}v$ is the value for which the energy is equal to the top of the Mexican hat potential. So we expect the transition from spontaneously broken symmetry and restoration of symmetry for an initial value $\phi_0 = \sqrt{2}v$.

An example for the evolution of the classical or mean field $\phi(t)$ is presented in Fig. 1. It is *not* typical, but it nicely illustrates the behavior very close to the phase transition, making apparent the coexistence of a broken symmetry and a symmetric minimum.

3.1 Stabilization

In the nonequilibrium evolution of models with spontaneous symmetry breaking both squared masses $\mathcal{M}^2_{1,2}$ can in general take negative values. In large-N

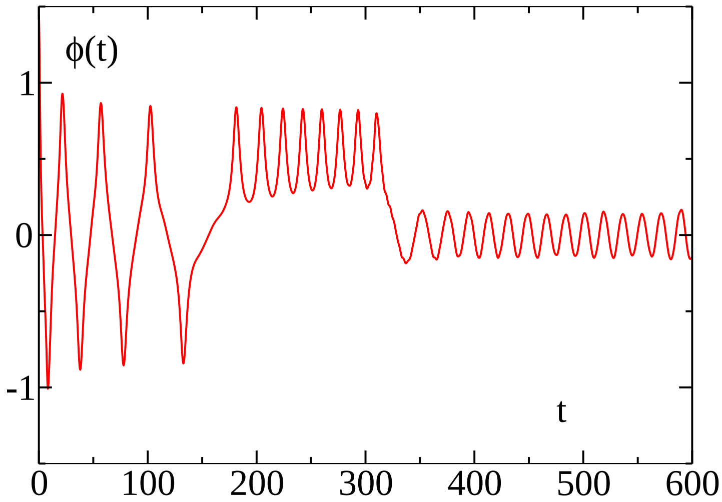

Figure 1: Time evolution of the classical field ϕ for $\phi_0 \simeq \sqrt{2}v$ (critical region). Parameters as above but $\phi_0 = 1.445v$.

dynamics it is well known [7, 8, 9, 10] that in this situation the fluctations increase exponentially and drive the squared masses back to positive values. This stabilizes the system dynamically and prevents the unphysical behavior of the one-loop approximation where an exponentially increasing amount of quantum energy, taken from the vacuum, is converted into classical one.

Our first essential observation is that this stabilization takes place as well at finite N in the Hartree approximation. The typical time evolution of the classical field and of the squared masses $\mathcal{M}^2_{1,2}$ is displayed in Fig. 2, based on the parameters $N = 4, \lambda = 1$, and $\phi_0 = 1.4v$. Both squared masses are seen to become negative at early times and to reach positive values at late times. The time where stabilization sets in is characterized by a strong increase of the quantum fluctuations.

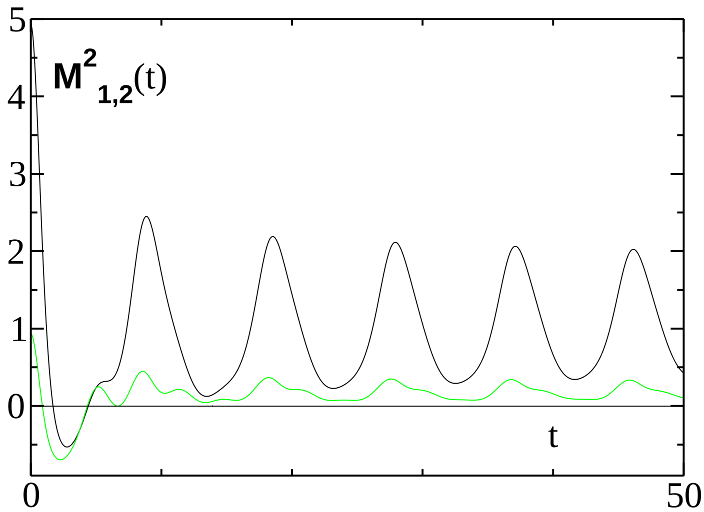

Figure 2: Time evolution of the masses $\mathcal{M}^2_{1,2}$. Parameters are $N = 4$, $\lambda = 1$, and $\phi_0 = 1.4v$. The upper curve is $\mathcal{M}^2_1$.

3.2 Late time behavior of the classical field

The value ϕ_∞ of the classical field, averaged at late times, is a function of the initial value ϕ_0, and therefore of the total energy. As ϕ_∞ is the value where the fields "settles" ultimately, it can be considered as an order parameter analogous to the vacuum expectation value $v(T)$ of finite temperature field theory. In the analysis of the broken symmetry phase in the large-N limit it was observed [9, 10] that ϕ_∞ has a specific dependence on the initial value ϕ_0, given at zero temperature by

$$\phi_\infty \simeq \left[\phi_0^2 \left(2v^2 - \phi_0^2\right)\right]^{1/4} \tag{12}$$

This is qualitatively the dependence one expects for the analogous function $v(T)$ in the case of a second order phase transition, with ϕ_0 replaced by T and v by T_0. The function $\phi_\infty(\phi_0)$, as determined numerically in our simulations, is displayed in Fig. 3. The shape is similar to the one found in the large-N case. It is somewhat difficult to analyze the behavior near the

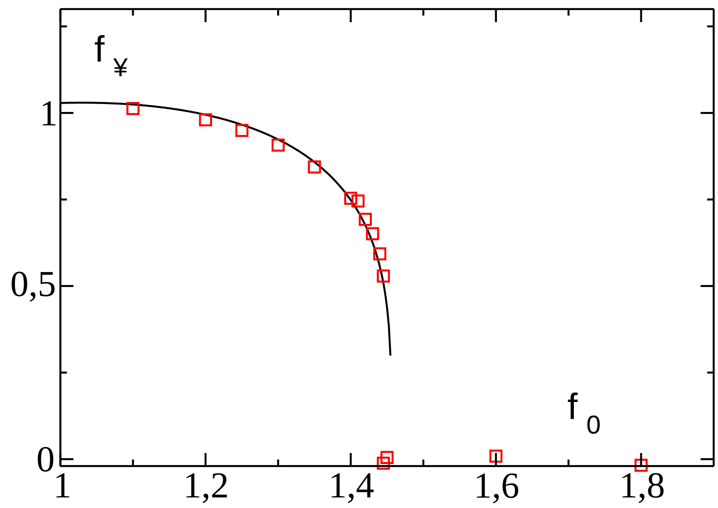

Figure 3: The order parameter $\phi_\infty(\phi_0)$ for $N = 4$ and $\lambda = 1$.

phase transition; for a second order transition the slope is infinite there, while for a first order transition one expects a discontinuity. Closer investigation gives evidence for the latter case. So the phase transition is (weakly) first order. The same is found, in this model and in the Hartree approximation, by analyzing $v(T)$ in thermal equilibrium. The same analogy between out-of-equilibrium and finite temperature quantum field theory holds in the limit $N \to \infty$ where a second order phase transition is found in both cases [2, 9, 10].

3.3 Late time behavior of the masses

Another set of variables characteristic for the phase structure of the $O(N)$ model in equilibrium are the mass scales or correlation lengths. In the broken symmetry phase we expect a vanishing pion mass and a nonzero value of the sigma mass. At a second order phase transition the sigma mass should go to zero, while for a first order transition it stays finite. Above the phase transition both masses should be equal.

The behavior of the various mass parameters is shown in Fig. 3. The

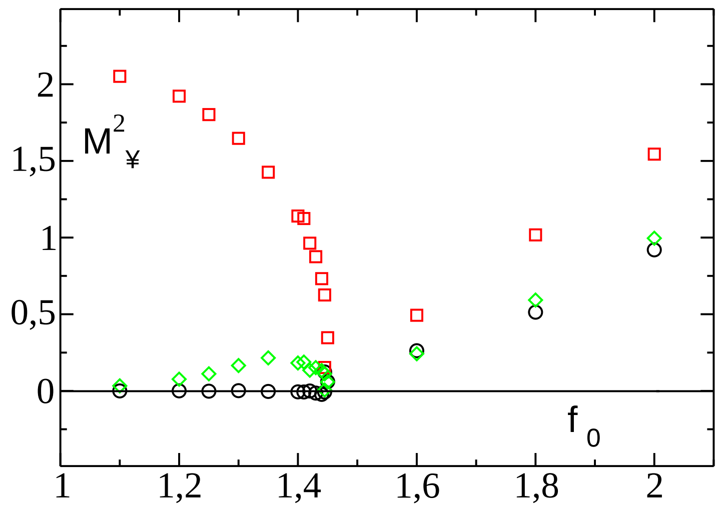

Figure 4: The various masses averaged at late times (t=300), as functions of the initial amplitude. Squares: $\mathcal{M}_1^2$, diamonds: $\mathcal{M}_2^2$, circles: $\mathcal{M}_{\rm cl}^2$. Parameters are $N = 4$ and $\lambda = 1$.

"pion" mass $\mathcal{M}_2$ does not vanish below the transition, but is small. There are arguments [15, 13] that the Goldstone mass should be rather identified with the mass of the classical field $\mathcal{M}_{\rm class}$ which indeed displays the expected behavior. The sigma mass drops near the phase transition but does not quite reach zero. This is in agreement with the assignment of a weakly first order phase transition. The fact that the masses do not become equal in the symmetric phase for $\phi_0 > v\sqrt{2}$ can be understood from the fact that the classical field continues to oscillate at late times, and so keeps in memory the explicit symmetry breaking due to the non-symmetric initial state.

3.4 Parametric resonance

In the large-N limit one of the most pronounced characteristic of the momentum spectra of the "pion" fluctuations is the occurrence of parametric resonance bands. These develop already in the early stage of evolution, before back-reaction sets in. Then the time dependence of the classical field is described by Jacobian elliptic functions, and the fluctuations are solutions of the Lamé equation. These solutions have been derived and discussed extensively in [7, 9].

Parametric resonance bands occur in the Hartree approximation as well, both in the broken and symmetric phases. We show a typical momentum spectrum in Fig. 5. What is shown there is the UV subtracted integrand

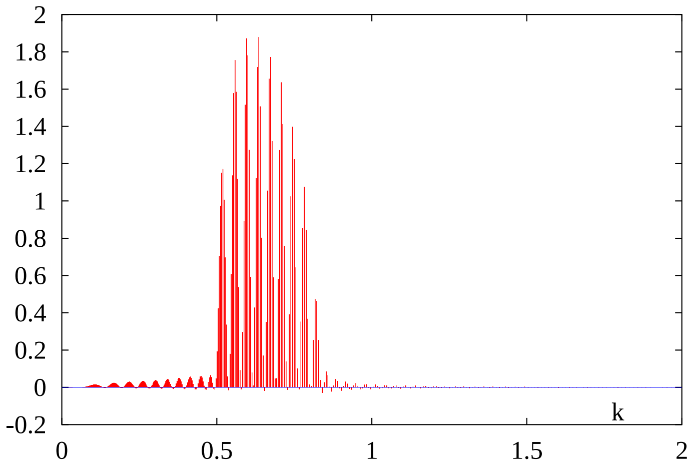

Figure 5: Spectrum of pion fluctuations in the broken symmetry phase. The integrand of the fluctuation integral as a function of k for $\lambda = 1$, $\phi_0 = 1.2v$ at $t = 100$.

of the fluctuation integral. While parametric resonance is welcome for an efficient transfer of classical into quantum energy, the presence of the resonance structure even at late times manifests the lack of thermalisation that is inherent in the approximation.

4 Conclusions

We have presented here an analysis of nonequilibrium dynamics in $O(N)$ models with spontaneous symmetry breaking at finite N in the Hartree approximation. We summarize our findings as follows:

(i) The back-reaction of the quantum fluctuations onto themselves leads, as in the large-N approximation, to a *stabilization* of the system, thus avoiding the catastrophic instability found in the one-loop approximation.

(ii) The dependence on the initial conditions displays an expected *phase transition* between a regime with spontaneous symmetry breaking and a symmetric phase.

(iii) The features of this phase transition are analogous to those found in quantum field theory at finite temperature, in the same approximation. The phase transition is weakly first order, while it is second order in the large-N limit.

(iv) The system does not thermalize in this approximation. Thermalization has been found to occur in higher approximations. These include the sunset and higher order diagrams, incorporating direct rescattering of the quantum fluctuations.

References

[1] see, e.g.: A. D. Linde, Phys. Rept. **333** (2000) 575.

[2] S. Coleman, R. Jackiw and H. D. Politzer, Phys. Rev. D **10**, 2491 (1974). L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974). W. A. Bardeen and M. Moshe, Phys. Rev. D **28**, 1372 (1983); Phys. Rev. D **34**, 1229 (1986).

[3] see e.g. E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics*, Pergamon Press, Oxford 1981, Chapter X.

[4] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).

[5] E. Calzetta and B. L. Hu, Phys. Rev. D **35**, 495 (1987); Phys. Rev. D **37**, 2878 (1988).

[6] H. Verschelde and M. Coppens, Phys. Lett. B **287**, 133 (1992); M. Coppens and H. Verschelde, Z. Phys. C **58**, 319 (1993).

[7] D. Boyanovsky, H. J. de Vega, R. Holman and J. F. Salgado, Phys. Rev. D **54**, 7570 (1996) [arXiv:hep-ph- 9608205].

[8] F. Cooper, S. Habib, Y. Kluger and E. Mottola, Phys. Rev. D **55**, 6471 (1997) [arXiv:hep-ph- 9610345].

[9] D. Boyanovsky, H. J. de Vega, R. Holman and J. Salgado, Phys. Rev. D **59**, 125009 (1999) [arXiv:hep-ph- 9811273].

[10] J. Baacke and K. Heitmann, Phys. Rev. D **62**, 105022 (2000) [arXiv:hep-ph- 0003317].

[11] J. Berges, Nucl. Phys. A **699**, 847 (2002) [arXiv:hep-ph- 0105311].

[12] B. Mihaila, F. Cooper and J. F. Dawson, Phys. Rev. D **63**, 096003 (2001) [arXiv:hep-ph- 0006254]. F. Cooper, J. F. Dawson and B. Mihaila, arXiv:hep-ph- 0209051.

[13] J. Baacke and S. Michalski, Phys. Rev. D **65**, 065019 (2002) [arXiv:hep-ph- 0109137].

[14] J. Baacke and A. Heinen, arXiv:hep-ph- 0212312.

[15] Y. Nemoto, K. Naito and M. Oka, Eur. Phys. J. A **9**, 245 (2000) [arXiv:hep-ph- 9911431]; H. Verschelde and J. De Pessemier, Eur. Phys. J. C **22**, 771 (2002) [arXiv:hep-th- 0009241].

[16] J. Baacke, K. Heitmann and C. Patzold, Phys. Rev. D **55**, 2320 (1997) [arXiv:hep-th- 9608006].