Ground State in Gluodynamics and $q\bar{q}$ Potential

K. Zarembo*†

Department of Physics and Astronomy,

University of British Columbia,

6224 Agricultural Road, Vancouver, B.C. Canada V6T 1Z1

Abstract

Static color charges in Yang-Mills theory are considered in the Schrödinger picture. Stationary states containing color sources, interquark potential and confinement criterion are discussed within the framework based on integration over gauge transformations which projects the vacuum wave functional on the space of physical, gauge-invariant states.

1 Introduction

Usually the potential of interaction between static quark and antiquark is extracted from Wilson loop expectation value. A Wilson loop is essentially non-local operator, since it describes the process developing in time. On the other hand, static charges are point-like and it might seem that in the Schrödinger picture, when stationary states are considered, they can be described by some local operators. Of course, such operators can not be local in terms of the original gluon fields, but it is reasonable to look for auxiliary

*Permanent address: Institute of Theoretical and Experimental Physics, B. Cheremushkinskaya 25, 117259 Moscow, Russia

†E-mail: zarembo@itep.ru/@theory.physics.ubc.ca

variables, in which color charge creation operators can be local. The best candidates for such variables are pure-gauge components of the gluon fields.

Longitudinal gluons are unphysical in Yang-Mills theory, but, once color charges are considered, longitudinal degrees of freedom do not decouple completely, since the electric field of a static charge is longitudinal and longitudinal gluons are responsible for color electric forces. In this respect, it is useful to introduce gauge degrees of freedom as independent variables from the outset. Such kind of variables naturally appears if the gauge invariance of physical states is imposed by averaging over gauge transformations. An approach to the ground state in Yang-Mills theory based on this procedure was advocated in Refs. [1, 2], where it was proposed to find an approximate vacuum wave functional from variational principle applied to the simplest, Guassian variational ansatz. Averaging over gauge transformations allows to make trial wave functional exactly gauge invariant. It also allows to construct easily the states describing static color sources and to formulate a relatively simple criterion of confinement in the Hamiltonian framework [3, 4].

2 Vacuum sector

The ground state in gluodynamics satisfies Schrödinger equation

$$H_{\rm YM}\Psi = E_{\rm vac}\Psi, \tag{2.1}$$

where $H_{\rm YM}$ is the Hamiltonian of $SU(N)$ gauge theory (in $A_0 = 0$ gauge):

$$H_{\rm YM} = \int d^3x\,\left(\frac{1}{2}\,E_i^A E_i^A + \frac{1}{4}\,F_{ij}^A F_{ij}^A\right). \tag{2.2}$$

When the wave function is a functional of the gauge potentials, the electric fields act as variational derivatives:

$$E_i^A = -i\,\frac{\delta}{\delta A_i^A}. \tag{2.3}$$

The Hamiltonian (2.2) commutes with operators $D_i E_i^A(x)$, and, apart from the Schrödinger equation, the physical states are subject to the Gauss' law constraint:

$$D_i E_i^A\,\Psi = 0. \tag{2.4}$$

The Gauss' law represents in the infinitesimal form the gauge invariance of the physical states:

$$\Psi[A^{\Omega}] = \Psi[A], \qquad A_i^{\Omega} = \Omega^{\dagger} \left(A_i + \frac{1}{g}\,\partial_i \right) \Omega. \tag{2.5}$$

There are many ways to solve the Gauss' law constraint – to use gauge-invariant variables [5], to fix the residual gauge freedom [6], or to enforce gauge invariance projecting the wave function on the physical subspace [1, 3, 7, 2, 4]. The projection operator has an elegant path-integral representation as an averaging over all gauge transformations:

$$\Psi[A] = \int [DU]\ \mathrm{e}^{\,-S[A^U]}, \tag{2.6}$$

This representation is well known in Yang-Mills theory at finite temperature [8] and was used to find an approximate ground state of the Yang-Mills theory by variational method [1, 2]. In a sense, eq. (2.6) is a general solution of the Gauss' law, since any gauge-invariant functional can be represented in this form by an appropriate choice of the "bare" state $\exp(-S[A])$.

3 Static charges

To introduce static charges, we couple Yang-Mills fields to infinitely heavy fermions. By infinitely heavy we infer fermions without kinetic term in the Hamiltonian:

$$H = \int d^3x \left(\frac{1}{2}\, E_i^A E_i^A + \frac{1}{4}\, F_{ij}^A F_{ij}^A + M\bar{\psi}\psi \right). \tag{3.1}$$

The fermion operators obey anticommutation relations

$$\left\{ \psi_{\alpha}^{a}(x), \psi_{\beta b}^{\dagger}(y) \right\} = \delta_{\alpha\beta}\delta^{a}{}_{b}\delta(x-y), \tag{3.2}$$

where a, b and α, β are color and spinor indices, respectively. The fermion fields transform in the fundamental representation of $SU(N)$:

$$\psi^U = U^{\dagger}\psi, \qquad \psi^{\dagger\,U} = \psi^{\dagger} U. \tag{3.3}$$

The ground state of the Hamiltonian (3.1) can be easily constructed, because H obeys simple commutation relations with fermion operators:

$$
\begin{aligned}
[H, \psi_\alpha(x)] &= \pm M\psi_\alpha(x), \qquad \begin{cases} +, & \alpha = 3,4 \\ -, & \alpha = 1,2 \end{cases} && (3.4) \\
\left[H, \psi^\dagger_\alpha(x)\right] &= \pm M\psi^\dagger_\alpha(x), \qquad \begin{cases} +, & \alpha = 1,2 \\ -, & \alpha = 3,4 \end{cases} . && (3.5)
\end{aligned}
$$

These relations are written in the basis in which γ^0 is diagonal. It is clear that $\psi_{1,2}(x), \psi^\dagger_{3,4}(x)$ are annihilation operators and the fermion vacuum is defined by equations

$$
\begin{aligned}
\psi_{1,2}(x)|0\rangle &= 0, \\
\psi^\dagger_{3,4}(x)|0\rangle &= 0. \qquad (3.6)
\end{aligned}
$$

The fermion vacuum is gauge invariant by itself, and the ground state of the Hamiltonian (3.1) is described by the wave function

$$
\Psi_{\text{vac}} = \Psi[A]|0\rangle = \int [DU] \; \mathrm{e}^{-S[A^U]} \, |0\rangle, \qquad (3.7)
$$

where $\Psi[A]$ is the vacuum wave functional in gluodynamics – the lowest-energy solution of the Schrödinger equation (2.1).

Operators $\psi_{3,4}(x), \psi^\dagger_{1,2}(x)$ create excited fermion states which contain static color charges. These states are not gauge invariant and, thus, it is necessary to project them on the physical subspace. Again, projection can be realized as averaging over gauge transformations. Therefore, physical states are obtained by acting of gauge transformed operators on the fermion vacuum with subsequent integration over the gauge group. This procedure can be considered as a kind of dressing of the bare fermions by their static color-electric fields. The dressed operators are gauge-invariant – the gauge indices are replaced by global color ones [4], which we mark by prime:

$$
\begin{aligned}
\psi^{U\,a'}_\alpha(y) &= U^{\dagger\,a'}{}_a(y)\psi^a_\alpha(y), \\
\psi^{\dagger\,U}_{\beta b'}(x) &= \psi^\dagger_{\beta b}(x) U^b{}_{b'}(x), \qquad (3.8)
\end{aligned}
$$

where $\alpha = 3,4$ and $\beta = 1,2$ (below spinor indices are omitted for brevity). The simplest example of a color-singlet state containing static charges is the one with meson quantum numbers:

$$\Psi_M(x,y) = \int [DU]\ \mathrm{e}^{-S[A^U]}\, \psi^{\dagger\,U}_{a'}(x)\psi^{U\,a'}(y)|0\rangle = \Psi^a{}_b[A;x,y]\,\psi^\dagger_a(x)\psi^b(y)|0\rangle, \tag{3.9}$$

where the gluonic part of the wave function,

$$\Psi^a{}_b[A;x,y] = \int [DU]\ \mathrm{e}^{-S[A^U]}\, U^a{}_{a'}(x) U^{\dagger\,a'}{}_b(y) \tag{3.10}$$

transforms in the representation $\bar{N}\otimes N$ of the gauge group in order to compensate gauge transformations of the fermions.

The above construction, in fact, generalizes the dressing of electron operators proposed by Dirac [9]:

$$\begin{aligned} \psi^{(*)}_\alpha(y) &= \mathrm{e}^{-ieV(y)}\psi_\alpha(y), \\ \psi^{(*)\,\dagger}_\beta(x) &= \mathrm{e}^{ieV(x)}\psi^\dagger_\beta(x), \end{aligned} \tag{3.11}$$

, where

$$V(x) = \int d^3y\,\frac{1}{-\partial^2}(x-y)\partial_i A_i(y). \tag{3.12}$$

Dressed operators (3.11) possess a number of important properties. They are gauge-invariant and create eigenstates of the free electro-magnetic Hamiltonian. They also play an important role in QED eliminating IR divergencies related to emission of soft photons [10, 11]. Different non-Abelian generalizations of the operators (3.11) were proposed [10, 12].

It is not difficult to reproduce the dressing (3.11) from the representation based on averaging over gauge transformations. In Abelian theory $U = \mathrm{e}^{ie\varphi}$, $A^U_i = A_i + \partial_i\varphi$ and the action $S[A]$ is quadratic; for simplicity we choose the diagonal quadratic form. Then the integration over gauge transformations in (3.10) is Gaussian and, for example, the state (3.9) describing two charges of opposite sign has the form:

$$\begin{aligned} \Psi[A;x,y] &= \int [d\varphi]\ \exp\left[-\frac{1}{2}\left(A_i+\partial_i\varphi\right)K\left(A_i+\partial_i\varphi\right)\right]\mathrm{e}^{ie\varphi(x)}\,\mathrm{e}^{-ie\varphi(y)} \\ &= C\ \mathrm{e}^{ieV(x)}\,\mathrm{e}^{-ieV(y)}\exp\left(-\frac{1}{2}A^\perp_i K A^\perp_i\right), \end{aligned} \tag{3.13}$$

where C is a field-independent constant and $A^\perp_i$ denotes transversal part of the gauge potentials: $A^\perp_i = (\delta_{ij} - \partial_i\partial_j/\partial^2)A_j$. As follows from conformal symmetry, the coefficient function K is equal to $|p|$ in the momentum space. Then the last factor in (3.13) is nothing but the vacuum wave functional for free electro-magnetic field. The first two factors reproduce Dirac dressing operators. For a generic charged state the operator ψ^U will be replaced by $\psi^{(*)}$ after averaging over gauge transformations due to the Gaussian nature of the integration over the Abelian gauge group.

4 Interquark potential

The energy of the state (3.9) after obvious subtractions determines $q\bar{q}$ interaction potential:

$$\frac{\langle \Psi_M | H | \Psi_M \rangle}{\langle \Psi_M | \Psi_M \rangle} = 2M + E_{\text{vac}} + V(x-y). \tag{4.1}$$

Matrix elements of gauge-invariant states, like those entering eq. (4.1), are proportional to the volume of the gauge group. The representation (2.6) allows to get rid of this infinite factor easily [1]. For example, the norm of the vacuum state is

$$\langle \Psi | \Psi \rangle = \int [DU][DU'][dA] \; \mathrm{e}^{-S[A^U]-S[A^{U'}]} = \text{const} \int [DU][dA] \; \mathrm{e}^{-S[A^U]-S[A]}. \tag{4.2}$$

This expression can be viewed as a partition function of a three-dimensional statistical system. Matrix elements of the Hamiltonian are determined by a linear response of this system on perturbation by the operator

$$R[A] = \int d^3x \left(\frac{1}{2} \frac{\delta^2 S[A]}{\delta A_i^A \delta A_i^A} - \frac{1}{2} \frac{\delta S[A]}{\delta A_i^A} \frac{\delta S[A]}{\delta A_i^A} + \frac{1}{4} F_{ij}^A F_{ij}^A \right), \tag{4.3}$$

which, essentially, is the Hamiltonian. So, defining the partition function

$$Z = \int [DU][dA] \; \mathrm{e}^{-S[A^U]-S[A]+\frac{\lambda}{2} R[A^U]+\frac{\lambda}{2} R[A]} \tag{4.4}$$

we express, for example, the vacuum energy in gluodynamics as a derivative of the free energy of the statistical system (4.4) with respect to λ:

$$E_{\text{vac}} = \frac{\langle \Psi | H_{\text{YM}} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \left. \frac{\partial}{\partial \lambda} \ln Z \right|_{\lambda=0}. \tag{4.5}$$

The quark-antiquark potential can be represented in the form [3, 4]:

$$V(x-y)=\frac{\partial}{\partial\lambda}\ln\left\langle\,\mathrm{tr}\,U(x)U^{\dagger}(y)\right\rangle\Big|_{\lambda=0}\,, \tag{4.6}$$

where averaging is defined by eq. (4.4). The confining behavior of the $q\bar{q}$ potential is rather natural from the point of view of this representation. Really, the correlation function in eq. (4.6) must decrease at infinity. So, its logarithm, the derivative of which determines the potential, always increases. This does not mean that the potential is always rising at infinity. If the correlator of gauge rotations increases as a power of distance with fixed exponent, the potential decreases. However, if a mass gap is generated and the correlator falls exponentially, the potential grows linearly:

$$\left\langle\,\mathrm{tr}\,U(x)U^{\dagger}(y)\right\rangle\propto\,\mathrm{e}^{\,-mr},\quad r=|x-y|\to\infty, \tag{4.7}$$

$$V(r)=\sigma r+\ldots, \tag{4.8}$$

with the string tension determined by the derivative of the mass with respect to λ:

$$\sigma=-\left.\frac{\partial m}{\partial\lambda}\right|_{\lambda=0}. \tag{4.9}$$

Hence, the confinement arises due to generation of a mass gap in the averaging over gauge transformations. More presizely, the confining potential is generated if there is a nonzero linear response of the mass gap on the perturbation by the Hamiltonian in eq. (4.4).

At short distances the potential can be calculated perturbatively. The free-gluon (zeroth order perturbative) wave functional is Gaussian – the action S is quadratic:

$$S[A]=\frac{1}{2}\,A_i^AKA_i^A+\ldots. \tag{4.10}$$

We do not specify the form of the kernel K, since it drops from the final answer. The matrices of gauge transformations are expanded as $U=\mathrm{e}^{\,g\Phi}=1+g\Phi^AT^A+\ldots$, and $A_i^U=A_i+\partial_i\Phi+\ldots$. The gauge potentials can be integrated out in this approximation by the change of variables: $A_i=\bar{A}_i-\partial_i\Phi/2$, and the partition function (4.4) takes the form

$$Z=\mathrm{const}\int[d\Phi]\,\exp\left[-\frac{1}{4}\,\partial_i\Phi^A\left(K+\frac{\lambda}{2}\,K^2\right)\partial_i\Phi^A\right]. \tag{4.11}$$

For the correlator which defines the interquark potential we get:

$$\begin{aligned}\left\langle \operatorname{tr} U(x)U^\dagger(y)\right\rangle &= N+\frac{g^2}{2}\left\langle \Phi^A(x)\Phi^A(y)-\frac{1}{2}\,\Phi^A(x)\Phi^A(x)\right.\\ &\quad \left.-\frac{1}{2}\,\Phi^A(y)\Phi^A(y)\right\rangle+O(g^4)\\ &= N+\frac{g^2(N^2-1)}{2}\Big(D(x-y)-D(0)\Big)+O(g^4),\end{aligned}$$

where D is the propagator of the field Φ: $D^{-1}=-\partial^2(K+\lambda K^2/2)/2$. The equation (4.6) now yields the Coulomb potential with correct coefficient:

$$\begin{aligned}V(x-y) &= -\frac{g^2(N^2-1)}{2N}\left(\frac{1}{-\partial^2}(x-y)-\frac{1}{-\partial^2}(0)\right)\\ &= -\frac{g^2(N^2-1)}{8N\pi r}+\text{self-energy}.\end{aligned} \tag{4.12}$$

The large-distance asymptotics of the potential is determined by infrared structure of the vacuum wave functional, which, generally speaking, is not known. For illustrative purposes we present here a calculation of the string tension under particular assumptions about the dependence of the wave functional on long-wavelength fields [3].

The starting point is derivative expansion, which probably is justified at large distances. The main assumption is that the derivative expansion of the kernel K in eq. (4.10) starts from the constant term (in momentum representation): $K(p)=M+O(p^2)$. This assumption is compatible with variational estimates of Ref. [1]; however, arguments in favor of $M=0$ also exist [13]. If we accept that $M\neq 0$, cubic and higher terms in $S[A]$, as well as the magnetic part of the Hamiltonian, are irrelevant being of higher order in derivatives. Thus, the action in (4.4) takes the form of a local expression quadratic in gauge fields:

$$Z=\int [dA]\,[DU]\,\exp\left[-\frac{1}{2}\left(M+\frac{\lambda}{2}\,M^2\right)\int d^3x\,\left(A_i^AA_i^A+A_i^{U\,A}A_i^{U\,A}\right)\right]. \tag{4.13}$$

The integration over gauge potentials can be eliminated by the change of variables $A_i=\bar{A}_i-\partial_iUU^\dagger/2g$. The remaining path integral over gauge transformations is the partition function of three-dimensional principal chiral field with local action:

$$Z = \int [dA]\,[DU]\,\exp\left[-\frac{1}{2g^2}\left(M + \frac{\lambda}{2}\,M^2\right)\int d^3x\ \mathrm{tr}\,\partial_i U^\dagger \partial_i U\right]. \tag{4.14}$$

Principal chiral field is expected to produce a mass gap. The simplest way to see it is to consider U and $U^\dagger$ as independent complex matrices and to impose the unitarity condition introducing a Lagrange multiplyer. If the Lagrange multiplyer acquires non-zero vacuum expectation value, the field U has massive propagator. Following Ref. [1], we use mean field approximation to determine the mass gap. The gap equation follows from unitarity condition $\left\langle UU^\dagger \right\rangle = \mathbf{1}$:

$$\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{p^2+m^2} = \frac{1}{2g^2N}\left(M + \frac{\lambda}{2}\,M^2\right). \tag{4.15}$$

Differentiating this equation in λ we get, according to (4.9),

$$\sigma = -\frac{\partial m}{\partial \lambda} = \frac{\pi M^2}{g^2 N} = \frac{M^2}{4\alpha_s N}. \tag{4.16}$$

This relation is obtained within the effective low-energy theory. Hence, the coupling here is not the running one; it is fixed on the typical energy scale: $\alpha_s \equiv \alpha_s(M)$. It is worth mentioning that the corrections to the zeroth order of derivative expansion and to the mean field approximation are parametrically not suppressed and can only be numerically small [1].

5 Discussion

The averaging over gauge transformations appears to be very convenient for consideration of static color charges in the Schrödinger picture. It allows to formulate rather simple criterion of confinement. But originally integration over the gauge group was proposed in the context of the variational approach [1]. In principle, variational calculations can be done analytically for the simplest Gaussian ansatz projected on the space of gauge-invariant states. The main problem in this approach is correct UV renormalization. If UV divergencies are not removed by standard counterterms, a variational approximation looses sense. The UV structure of the vacuum wave functional was discussed from different point of views in Refs. [14].

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References

[1] I.I. Kogan and A. Kovner, Phys. Rev. D52 (1995) 3719, hep-th/9408081.

[2] D. Diakonov, hep-th/9805137.

[3] K. Zarembo, Phys. Lett. B421 (1998) 325, hep-th/9710235.

[4] K. Zarembo, hep-th/9806150.

[5] J. Goldstone and R. Jackiw, Phys. Lett. B74 (1978) 81; Yu.A. Simonov, Sov. J. Nucl. Phys. 41 (1985) 835 [Yad. Fiz. 41 (1985) 1311]; M. Bauer, D.Z. Freedman and P.E. Haagensen, Nucl. Phys. B428 (1994) 147, hep-th/9405028; P. E. Haagensen and K. Johnson, Nucl. Phys. B439 (1995) 597, hep-th/9408164; D. Karabali and V.P. Nair, Nucl. Phys. B464 (1996) 135, hep-th/9510157; P. E. Haagensen, K. Johnson and C. S. Lam, Nucl. Phys. B477 (1996) 273, hep-th/9511226; D. Karabali, C. Kim and V. P. Nair, hep-th/9804132.

[6] N.H. Christ and T.D. Lee, Phys. Rev. D22 (1980) 939; P. van Baal, hep-th/9511119; hep-th/9711070.

[7] C. Heinemann, C. Martin, D. Vautherin and E. Iancu, hep-th/9802036.

[8] A.M. Polyakov, Phys. Lett. B72 (1978) 477; *Gauge Fields and Strings* (Harwood Academic Publishers, 1987); L. Susskind, Phys. Rev. D20 (1979) 2610; B. Svetitsky, Phys. Rep. 132 (1986) 1.

[9] P.A.M. Dirac, Can. J. Phys. 33 (1955) 650; *The Principles of Quantum Mechanics* (Oxford, Clarendon Press, 1958).

[10] M. Lavelle and D. McMullan, Phys. Rep. 279 (1997) 1, hep-ph/9509344.

[11] E. Bagan, M. Lavelle and D. McMullan, Phys. Rev. D56 (1997) 3732, hep-th/9602083; D57 (1998) 4521, hep-th/9712080.

[12] P.E. Haagensen and K. Johnson, hep-th/9702204; M. Lavelle and D. McMullan, hep-th/9805013.

[13] J. Greensite, Nucl. Phys. B158 (1979) 469; Phys. Lett. B191 (1987) 431; J. Greensite and J. Iwasaki, Phys. Lett. B223 (1989) 207.

[14] W.E. Brown and I.I. Kogan, hep-th/9705136; K. Zarembo, Mod. Phys. Lett. A13 (1998) 1709, hep-th/9803237; 1795, hep-ph/9804276.