Absolute upper bounds on $\pi^0\pi^0$ elastic scattering amplitude at arbitrary energies

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Abstract

Absolute upper bounds on the modulus of $\pi^0\pi^0$ elastic scattering amplitude are obtained for arbitrary physical energies. These bounds are the consequence of analyticity, unitarity and crossing symmetry only.

One of the remarkable achievements of axiomatic quantum field theory is the derivation of absolute (numerical) bounds on $\pi^0\pi^0$ elastic scattering amplitude. These bounds are rigorous consequences of general principles of quantum field theory only. Precisely they are consequences of analyticity in s and t planes, crossing symmetry and unitarity. The first bound of this kind has been derived by Martin [1]. The improved bounds were obtained in paper of Łukashúk and Martin [2] and then in papers [3-8]. Above mentioned bounds are bounds in nonphysical region, where $\mid s \mid< 4m_\pi^2, \mid t \mid< 4m_\pi^2$ and $\mid u \mid< 4m_\pi^2$ or bounds on some integrals over low energies intervals. As it has been shown by A. Martin [1], such bounds, in fact, imply the restrictions on the physical amplitudes, precisely, on the integrals of the scattering amplitude over all physical region.

The distinguished role of $\pi^0\pi^0$ scattering follows from the full symmetry of its elastic scattering amplitude $F(s,t,u)$ in s, t and u-variables. According to the quark structure of hadrons $\pi^0\pi^0$ total cross section $\sigma_{tot}^{\pi^0\pi^0}$ has the same order as $\sigma_{tot}^{\pi N}$ and σ_{tot}^{NN}.

In our paper [9] absolute upper bounds on $\pi^0\pi^0$ total cross section at arbitrary physical energies have been derived. The present report is the continuation of this paper. Here we show that absolute bounds on $\mid F(s,t) \mid$ in the region: $\mid s \mid < 4\,m_\pi^2,\ 0 < t < 4\,m_\pi^2$ lead to the absolute bounds on $\mid F(s,t) \mid$ at physical s and the same t.

Precisely we obtain bounds on $\mid F(\bar{s},t) \mid$, where $\bar{s}$ is some point in the interval $(s, \nu\, s),\ \nu > 1$ and in other respects is arbitrary. The dependence of the obtained bounds on ν is rather weak, so we have reasonable bound even in case $\nu - 1 \ll 1$ (almost local bound).

Bounds on $\mid F(s,t) \mid$ at $s > 4\,m_\pi^2$ and $t > 0$ implies absolute bounds on $\mid F(s,t) \mid$ at physical s and t. At $t = 0$ this bound has the same form as Froissart-Martin bound [2, 10], but this "absolute Froissart-Martin bound" has not unknown constant and so is valid at finite energies.

To derive upper bound on $|F(s,t)|$ at arbitrary physical s and positive t inside Martin ellipse we use DR for the $lnF(s,t)$ (LDR). It is convenient to change s on $z \equiv (s - 2m^2 + t/2)^2$. Function $F(z,t)$ has only one right cut $(z_0, \infty),\ z_0 = (2m^2 + t/2)^2$ and DR has the simplest form:

$$F(z,t) = F(0,t) + \frac{z}{\pi} \int\limits_{z_0}^{\infty} \frac{A(z',t) d\,z'}{z'(z'-z)} \qquad (1),$$

where $A(z,t) \equiv ImF(z,t)$.

It is well known that according to the result of Jin and Martin [11] DR require only two subtractions if $t > 0$ and belongs to Martin ellipse. As $A(z,t) > 0$ at $0 \leq t < 4m^2$, then $F(z,t)$ has at least one zero for the same reason as at $t = 0$ (see e.g. [12]). As

$$F(z^*,t) = F^*(z,t),$$

then this zero must be in a real point.

If $F(z,t)$ has zero at point $\hat{z}$:

$$F(z,t) = (z - \hat{z})\, \varphi(z,t) \qquad (2),$$

then LDR have to be written for $\varphi(z,t)$:

$$ln \left| \frac{\varphi(z,t)}{\varphi(z_i,t)} \right| = \frac{z - z_i}{\pi} P \int_{z_0}^{\infty} \frac{\delta(z',t) d\,z'}{(z'-z)\,(z'-z_i)} \qquad (3),$$

where $\delta\,(z^{'},t)$ is the phase of function $\varphi\,(z',t)$, z_i belongs to the interval $(0,z_0)$. Let's point out that $ln\,\varphi\,(z^*,t) = (ln\,\varphi\,(z,t))^*$. In order to show this it is sufficient to show that $\delta\,(z^{'}-i\varepsilon,t) = -\delta\,(z^{'}+i\varepsilon,t)$. First, in accordance with positiveness of $A\,(z^{'},t)$ $\delta\,(z^{'},t)$ satisfies the following double inequality:

$$0 \leq \delta\,(z^{'},t) \leq \pi \tag{4}$$

From DR for $\varphi\,(z,t)$ we have:

$$\varphi\,(0,t) = \frac{1}{\pi}\int_{z_0}^{\infty}\frac{Im\varphi\,(z',t)d\,z^{'}}{z^{'}} > 0$$

So $\varphi(z_0,t) > 0$, $\delta\,(z_0,t) = 0$ and as $\varphi(z^*,t) = (\varphi\,(z,t))^*$, the necessary result is evident.

If $F\,(z,t)$ has no zeroes, then eq.(3) is valid for $lnF\,(z,t)$ as well. Only if $F\,(z_0,t) < 0$, $\delta\,(z^{'},t)$ must be substituted by $\tilde{\delta}\,(z^{'},t) \equiv \delta\,(z^{'},t) - \pi$. Evidently

$$-\pi \leq \tilde{\delta}\,(z^{'},t) \leq 0 \tag{4'}$$

To obtain the desirable bound in the same way both in case of existence and absence of zeroes of $F\,(z,t)$, let's consider function

$$\psi_\beta(z,t) = \mid z - z_0 \mid^\beta \varphi\,(z,t); \qquad \beta \in \mathbb{R} \qquad z > 0 \tag{5}$$

It can be checked that

$$ln\frac{z-z_0}{z_0-z_i} = (z_i - z)P\int_{z_0}^{\infty}\frac{d\,z^{'}}{(z^{'}-z_i)(z^{'}-z)} \qquad z_i \in (0,z_0) \tag{6}$$

Thus

$$ln\left|\frac{\psi_\beta(z,t)}{\psi_\beta(z_i,t)}\right| = (z-z_i)P\int_{z_0}^{\infty}\frac{d\,z^{'}\,\delta_\beta(z^{'})}{(z^{'}-z_i)(z^{'}-z)} \tag{7}$$

where

$$\delta_\beta(z) = \delta\,(z) - \beta\pi$$

So

$$-\beta\pi \leq \delta_\beta(z) \leq \pi(1-\beta) \tag{8}$$

Below for simplicity we consider the case $z \gg z_0$. As $t \leq 4m_\pi^2$, $z_0 \leq 16m_\pi^4$, then $z/z_0 > 100$, if $\sqrt{s} > 1Gev$. So, actually all energies higher then 1Gev satisfy our condition.

If $z \gg z_0$, then $\psi_1(z,t) \cong F(z,t)$ (we can suppose that $0 < \hat{z} < 4m_\pi^2$ as in this case obtained bound are found to be absolute).

If $F(z,t)$ has no zeroes and $F(z_0,t) > 0$, then LDR for $F(z,t)$ coincide with LDR for $\varphi(z,t)$.

Now let's choose some interval $(z, \mu z)$, $z > z_0$, $\mu > 1$.

To use efficiently eq.(7) let's integrate both parts of this equation with some positive normed weight function $\rho(z)$: $\int_z^{\mu z} \rho(z')dz' = 1$. Using the low of the mean in the left part of the equality and changing the order of integration in the right part of it, we obtain:

$$ln\left|\frac{\psi_\beta(\bar{z},t)}{\psi_\beta(z_i,t)}\right| = \frac{1}{\pi}\int_{z_0}^{\infty}\frac{\delta_\beta(z')}{z'-z_i}P\int_z^{\mu z}\frac{(z''-z_i)\,\rho(z'')\,dz''}{z'-z''} \tag{9}$$

Upper bound for $\psi_\beta(\bar{z},t)/\psi_\beta(z_i,t)$ follows from condition (8) and depends on the choice of function $\rho(z)$. We consider sufficiently wide class of functions $\rho(z)$, namely $\rho_\alpha(z) = C_\alpha\, z^{\alpha-1}$, $C_\alpha = \alpha/z^\alpha(\mu^\alpha - 1)$, $\alpha \in \mathbb{R}$.

We obtain explicit result at arbitrary α. It gives the possibility for any given μ to choose α in such a way, that resulting bound would be optimal.

Using condition $z \gg z_0$ we can rewrite eq.(9):

$$ln\left|\frac{\psi_\beta(\bar{z},t)}{\psi_\beta(z_i,t)}\right| = \frac{\alpha}{\pi(\mu^\alpha-1)}\int_{u_0}^{\infty}\frac{\delta(u)du}{u-u_i}\Phi_\alpha(u) \tag{10}$$

$$\Phi_\alpha(u) = \int_1^\mu \frac{y^\alpha dy}{y-u} \qquad u_0 = \frac{z_0}{z}; \qquad u_i = \frac{z_i}{z}.$$

It can be easily shown that for any α there exists such $\tilde{u}_\alpha$, that

$$\Phi_\alpha(\tilde{u}_\alpha) = 0,$$

$$\Phi_\alpha(u) < 0, \qquad u < \tilde{u}_\alpha;$$

$$\Phi_\alpha(u) > 0 \qquad u > \tilde{u}_\alpha$$

It is evident that $1 < \tilde{u}_\alpha < \mu$. So in accordance with condition (8):

$$ln\left|\frac{\psi_\beta(\bar{z},t)}{\psi_\beta(z_i,t)}\right| \leq (1-\beta)\,I_\alpha; \qquad \beta \leq 0 \tag{11}$$

$$ln\left|\frac{\psi_\beta\left(\bar{z},t\right)}{\psi_\beta\left(z_i,t\right)}\right| \leq \beta\,\tilde{I}_\alpha; \qquad \beta\geq 1 \tag{12}$$

$$ln\left|\frac{\psi_\beta\left(\bar{z},t\right)}{\psi_\beta\left(z_i,t\right)}\right| \leq \beta\tilde{I}_\alpha+\left(1-\beta\right)I_\alpha; \qquad 0\leq\beta\leq 1 \qquad (13),$$

where

$$I_\alpha=\frac{\alpha}{\mu^\alpha-1}\int_{\tilde{u}}^{\infty}\frac{d\,u}{u}\Phi_\alpha\left(u\right) \tag{14}$$

$$\tilde{I}_\alpha=\frac{\alpha}{\mu^\alpha-1}\int_{u_0}^{\tilde{u}}\frac{d\,u}{u-u_i}\Phi_\alpha\left(u\right) \tag{15}$$

I_α and $\tilde{I}_\alpha$ can be easily obtained if $\alpha\neq 0$.

Actually, changing the order of integrations we can show that

$$I_\alpha=ln\,\tilde{u}_\alpha-\frac{\alpha}{\mu^\alpha-1}\int_1^\mu y^{\alpha-1}ln\left|\tilde{u}_\alpha-y\right|d\,y \tag{16}$$

The last integral can be calculated by using integration by parts and taking into account that $\Phi_\alpha\left(\tilde{u}_\alpha\right)=0$.

Final result is:

$$I_\alpha=ln\,\frac{\tilde{u}_\alpha}{\mu-\tilde{u}_\alpha}+\frac{1}{\mu^\alpha-1}ln\,\frac{\tilde{u}_\alpha-1}{\mu-\tilde{u}_\alpha} \tag{17}$$

By the same way it is easy to show that

$$\tilde{I}_\alpha=I_\alpha+ln\,\frac{z}{z_0-z_i}+\frac{\mu^\alpha\,ln\,\mu}{\mu^\alpha-1}-\frac{1}{\alpha} \tag{18}$$

I_0 can be estimated directly or as a $lim_{\alpha\to 0}I_\alpha$.

We make use of a second possibility as in this way we obtain also approximate expression for I_α in the case $\mu^\alpha-1\ll 1$.

If $y^\alpha\cong 1+\alpha\,ln\,y,\ 1\leq y\leq\mu$, then

$$\frac{1}{\alpha}\,ln\,\frac{\tilde{u}_\alpha-1}{\mu-\tilde{u}_\alpha}\cong P\int_1^\mu\frac{ln\,y\,d\,y}{\tilde{u}_\alpha-y} \tag{19}$$

Calculating the last integral we obtain:

$$ln\,\mu \lim_{\alpha\to 0} \frac{1}{\alpha}\, ln\, \frac{\tilde{u}_\alpha - 1}{\mu - \tilde{u}_\alpha} = \begin{array}{l} \frac{\pi^2}{3} - \frac{1}{2}\left(ln^{\,2}\mu + ln^{\,2}\tilde{u}_\alpha\right) + ln\,(\mu - \tilde{u}_\alpha) - \\ -Li_2\left(\frac{1}{\tilde{u}_\alpha}\right) - Li_2\left(\frac{\tilde{u}_\alpha}{\mu}\right), \end{array} \tag{20}$$

where

$$Li_2(x) \equiv \sum_1^\infty \frac{1}{n^2}\, x^{\,n}; \qquad 0 < x < 1$$

In accordance with eqs. (15) and (18):

$$I_0 = \frac{1}{ln\,\mu}\left[\frac{\pi^2}{3} - \frac{1}{2}\, ln^{\,2}\frac{\mu}{\tilde{u}_0} - Li_2\left(\frac{1}{\tilde{u}_0}\right) - Li_2\left(\frac{\tilde{u}_0}{\mu}\right)\right] \tag{21}$$

It is easy to see that

$$\tilde{u}_0 = \frac{1+\mu}{2} \tag{22}$$

Eq.(21) (with the evident substitution $\tilde{u}_0 \to \tilde{u}_\alpha$) is also an approximate expression for I_α in case $\mu^\alpha - 1 \ll 1$.

In accordance with eq.(16)

$$\tilde{I}_0 = I_0 + ln\,\frac{\sqrt{\mu}\, z}{z_0 - z_i} \tag{23}$$

and

$$\tilde{I}_\alpha = I_\alpha + ln\,\frac{\sqrt{\mu}\, z}{z_0 - z_i} + \frac{\alpha\, ln^{\,2}\mu}{12} \qquad \text{if} \qquad \alpha\, ln\,\mu \ll 1 \tag{24}$$

Let's describe now how to calculate $\tilde{u}_\alpha$. $\tilde{u}_\alpha$ is determined by the condition $\Phi_\alpha\,(\tilde{u}_\alpha) =$ (see eq. (10)). This condition can be written as equality:

$$\int_1^{\tilde{u}_\alpha} \frac{y^\alpha\, d\,y}{\tilde{u}_\alpha - y} = \int_{\tilde{u}_\alpha}^{\mu} \frac{y^\alpha\, d\,y}{y - \tilde{u}_\alpha} \tag{25}$$

Calculating integrals in both sides of eq. (21) we come to the equation for $\tilde{u}_\alpha$:

$$
\begin{aligned}
&\tilde{u}_\alpha \sum_{n=0}^{\infty} \frac{(2\alpha+1)}{(n-\alpha)(n+\alpha+1)} = \\
&\quad = \mu^{\,\alpha} \sum_{n=0}^{\infty} \frac{1}{(n-\alpha)} \left(\frac{\tilde{u}_\alpha}{\mu}\right)^n - \sum_{n=0}^{\infty} \frac{1}{(\tilde{u}_\alpha)^{n+1}(n+\alpha+1)}; \\
&\quad \alpha \notin \mathbf{Z}_{\pm}, \qquad \alpha \neq 0
\end{aligned}
\tag{26}
$$

If $\alpha \in \mathbf{Z}_{\underline{+}}$, then it is more simply to make computation directly.

After elementary calculation we obtain the equations:

$$
ln\, \frac{\tilde{u}_n}{\mu - \tilde{u}_n} = \sum_{1}^{n} \frac{1}{k} \left(\frac{\mu}{\tilde{u}_n}\right)^k + \sum_{n+1}^{\infty} \frac{1}{k\,(\tilde{u}_n)^k} \tag{27}
$$

$$
ln\, \frac{1}{\tilde{u}_{-n} - 1} = \sum_{1}^{n} \frac{1}{k} \,(\tilde{u}_{-n})^k + \sum_{n+1}^{\infty} \frac{1}{k} \left(\frac{\tilde{u}_{-n}}{\mu}\right)^k \tag{28}
$$

If $\mu \gg 1$, then infinite sums in (26) and (27) are unessential. For $\alpha = 0$ see eq.(20). It is easy to show that $\tilde{u}_{-\frac{1}{2}} = \sqrt{\mu}$, $\tilde{u}_{-1} = 2\mu/(\mu+1)$.

As we derived the explicit formulae for I_α and $\tilde{I}_\alpha$ and equation for $\tilde{u}_\alpha$, we can obtain by numerical calculations the optimal value of $\alpha \equiv \alpha_{opt}$ for any given μ.

In case $\alpha = -1$ we have very simple expression for $\tilde{I}_\alpha$:

$$
\tilde{I}_{-1} = ln\, \frac{z}{z_0 - z_i} + \ln\, \frac{2\,e\,\mu}{\mu - 1} \tag{18'}
$$

Now let's proceed to numerical bounds. We need to use some bounds at nonphysical region. To this end we use the bounds on $F\,(s,t,u)$ obtained by Lopez and Mennessier [7]:

$$
\begin{aligned}
|\ F\,(1,3,0)\ | &< 3.3 \\
|\ F\,(2,3,0)\ | &< 14.5
\end{aligned}
\tag{29}
$$

We put $m_\pi^2 = 1$, so $z_0 = \frac{49}{4}$. In our variables (z,t) these results mean:

$$
\begin{aligned}
|\ F\,(1/4,3)\ | &< 3.3 \\
|\ F\,(9/4,3)\ | &< 14.5
\end{aligned}
\tag{29'}
$$

So if $F(z,3)$ has no zero, then

$$| F(\bar{z}, 3) |< 0.275\, z\, C\,(\alpha, \mu) \qquad (30),$$

where

$$C\,(\alpha, \mu) = exp\left[I_\alpha + \frac{\mu^\alpha\, ln\, \mu}{\mu^\alpha - 1} - \frac{1}{\alpha}\right]$$

If $F\,(z, 3)$ has a zero, then it is easy to see that the weakest, e.g. absolute, bound corresponds the point of zero, which satisfies the following condition: $3.3\,(\hat{z} - 1/4)^{-1} = 14.5\,(9/4 - \hat{z})^{-1}$. So absolute bound for $\mid F\,(\bar{z}, 3) \mid$ is:

$$| F\,(\bar{z}, 3) |< 8.9\, z\, C\,(\alpha, \mu) \qquad (31)$$

For $\alpha = -1$ $C\,(\alpha, \mu) = \frac{2\,e\,\mu}{\mu - 1}$ (see eq.(18′)).

The absolute bound (31) implies the analogous bound on $A\,(\bar{z}, 3)$.

Let's point out that obtained bound for $\mid F\,(\bar{z}, t) \mid$ depends very weakly on the behavior of the scattering amplitude at $z^{'} > \mu z$. In fact it can be improved only by the factor $exp\,(-I_\alpha)$.

We can obtain for $A\,(\bar{z}, 3)$ bound stronger then for $\mid F\,(\bar{z}, t) \mid$. To this end we consider the behavior of $\delta\,(z^{'}, t)$ which leads to $max\, A\,(\hat{z}, t)$. In fact the behaviour of $\delta\,(z^{'}, t)$ corresponding $max\, |F\,(\bar{z}, t)|$ implies $A\,(\bar{z}, t) = 0$. Evidently we have to change $\delta\,(z^{'}, t)$ only in the interval $(z, \mu z)$. The necessary behaviour of $\delta\,(z^{'}, t)$ can be easily obtained by standard variational methods.

Let's point out that upper bound on $A\,(\bar{z}, t)$ can be derived directly from DR [9].

In conclusion let's see what absolute bound on $A\,(\bar{z}, 0)$ follows from the analogous bound on $A\,(\bar{z}, t)$ (for details see [9]). It is well known that at given $A\,(\bar{z}, t)$ $max\, A\,(\bar{z}, 0)$ is realized if partial amplitudes $a_l\,(z) = 1$ till some $l = L$ and then $a_l\,(z) = 0$. Thus the sum over $a_l\,(z)$ at $t > 0$ and $t = 0$ can be easily expressed in terms of L and L is determined by our absolute bound.

The final result is:

$$\sigma_{tot}^{max} \leq \frac{2\pi}{t}\, ln^2\,(Bs) \qquad (32),$$

where B is a known constant.

By the same way, using the standard methods, we can obtain the absolute bounds on $\mid F\,(s, t) \mid, t \leq 0$ and other quantities, for example on $a_l\,(s)$ at $l > L$.

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