Exact Results Versus Anomalies

P. I. Pronin and K. V. Stepanyantz
Moscow State University, Physics Faculty,
Department of Theoretical Physics.
117234*, Moscow, Russian Federation*

Abstract

We investigate nonperturbative effects in the N=1 SUSY Yang-Mills theory with matter using a relation between perturbative and exact anomalies as a starting point. We show, that the superpotential is always generated by instantons and should contain gluino condensate. The exact expression for the superpotential is found.

1 Introduction

One of the most interesting QFT problem is the investigation of nonperturbative dynamics. It is well known [1, 2], that except for perturbative corrections there is a series of instanton contributions. The first exact nonperturbative result was found in [3] for N=2 Yang-Mills theory in the constant field limit. It was checked, that its assimptotic reproduces one-instanton contribution correctly [4, 5].

Attempts to construct exact results for N=1 SUSY theories were made in [6, 7]. However, the corresponding results can not be considered as satisfactory, because for $N_f < N_c - 1$ they contradict to instanton calculations [6]. For $N_f \geq N_c$ the prove of agreement is not beautiful and natural, for $N_f = N_c$ again the Lagrange multiplier arises unnaturally. Moreover, Affleck-Dine-Seiberg results do not produce correct expressions for anomalies [8].

In principle, at the perturbative level the anomalies is correctly reproduced by the Veneziano-Yankielowitch effective Lagerangian [9], containing gluino condensate. However, this result is not applicable beyond the frames of perturbation theory.

The purpose of the present paper is to obtain true expression for the nonperturbative effective Lagrangian of N=1 SUSY Yang-Mills theory with matter. It should correctly produce anomalies, be in agreement with instanton calculations and have a structure at least similar to the one for N=2 case (because N=2 SUSY theories have also N=1 SUSY).

2 Supersymmetric Yang-Mills theories

The massless N=1 supersymmetric Yang-Mills theory with $SU(N_c)$ gauge group and N_f matter multiplets is described by the action

$$S = \frac{1}{16\pi}\mathrm{tr}\ \mathrm{Im}\left(\tau\int d^4x d^2\theta\ W^2\right) + \frac{1}{4}\int d^4x d^4\theta \sum_{A=1}^{N_f}\left(\phi^+_A e^{-2V}\phi^A + \tilde\phi^{+A}e^{2V}\tilde\phi_A\right) \tag{1}$$

where the matter superfields ϕ and $\tilde\phi$ belong to fundamental and antifundamental representations of the gauge group $SU(N_c)$.

In the massless case the action is invariant under the transformations

$$\begin{aligned} U(1)_x: \quad & W(\theta)\to e^{i\alpha}W(e^{-i\alpha\gamma_5}\theta);\\ & \phi(\theta)\to e^{ix\alpha}\phi(e^{-i\alpha\gamma_5}\theta); \qquad \tilde\phi(\theta)\to e^{ix\alpha}\tilde\phi(e^{-i\alpha\gamma_5}\theta). \end{aligned} \tag{2}$$

where x is an arbitrary constant.

It is easy to see, that this symmetry produces chiral transformations for fermions. In the perturbation theory the anomaly of correspondent current is

$$\begin{aligned} \partial_\mu J^\mu_x &= \Big(-N_f + N_c + xN_f\Big)\frac{1}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{tr}F_{\mu\nu}F_{\alpha\beta}\\ &= \Big(N_f - N_c - xN_f\Big)\frac{1}{16\pi^2}\mathrm{Im}\ \mathrm{tr}\int d^2\theta\ W^2. \end{aligned} \tag{3}$$

In particular, for $x=(N_f-N_c)/N_f$ we obtain anomaly free $U(1)_R$-symmetry.

3 Instanton contributions to anomalies and the structure of the effective potential

Now let us find the general structure of instanton corrections to superpotential and anomalies.

The effective Lagrangian can be split into the following parts [9]

$$L_{eff} = L_k + L_a + L_m, \tag{4}$$

where

$$\begin{aligned} L_k &= \int d^4\theta \; K(S, S^*, \Phi, \Phi^*); \\ L_a &= \mathrm{Re} \int d^2\theta \; w(S, \Phi) \end{aligned} \tag{5}$$

and $S \equiv \mathrm{tr} W^2$

Here L_k denotes kinetic terms, that do not contribute to the anomaly, L_a is a holomorphic part of the superpotential and L_m is a mass term. Below we will consider only massless case ($L_m = 0$). Therefore, the only nontrivial contribution to anomalies comes from L_a and it is the only part, that we are able to investigate. (Our method can not give any information about a possible kinetic term.)

Performing $U(1)_x$ transformation in the effective action we obtain

$$\langle \partial_\mu J^\mu_x \rangle = -\frac{\partial \Gamma}{\partial \alpha} = -\mathrm{Im} \int d^2\theta \Big(2w - 2\frac{\partial w}{\partial S} S - x \frac{\partial w}{\partial v} v \Big), \tag{6}$$

where we substituted ϕ and $\tilde{\phi}$ by their vacuum expectation values v.

On the other hand, the anomaly can be found from the transformation law of the collective coordinate measure. Comparing the result with 6 we obtain [8]

$$w^{(n)} = W^2 g_n \left(\frac{v^3}{W^2} \right) \left(\frac{\Lambda}{v} \right)^{n(3N_c - N_f)} = S g_n \left(\frac{v^3}{S} \right) \left(\frac{\Lambda}{v} \right)^{n(3N_c - N_f)} \tag{7}$$

where g_n is an arbitrary function. Its explicit form will be found below ¿from the relation between perturbative and exact anomalies.

Of course, (7) is in a complete agreement with dimensional arguments and does not depend on the particular choice of symmetry (i.e. x).

4 The relation between perturbative and exact anomalies

So, there are nonperturbative corrections to anomalies. Nevertheless, there is a way to treat nonperturbative expressions, anomalies, in particular. It is exact results. For example, starting with the Seiberg-Witten expressions [3] it is possible to find R-anomaly [10, 11]. The result is the following: Exact anomaly is the vacuum expectation value of the perturbative one. This result is not unexpected. Really, performing, for example, chiral transformation in the generating functional we have

$$0 = \frac{1}{Z}\frac{\delta Z}{\delta\alpha}\Big|_{\alpha=0} = \frac{1}{Z}\frac{\delta}{\delta\alpha}\int DAD\bar{\psi}'D\psi'\exp\Big(iS - \partial_\mu\alpha j_5^\mu\Big)\Big|_{\alpha=0}$$
$$= \frac{1}{Z}\frac{\delta}{\delta\alpha}\int DAD\bar{\psi}D\psi\exp\Big(iS - \partial_\mu\alpha j_5^\mu - \alpha\mathbf{A}\Big)\Big|_{\alpha=0} = \langle\partial_\mu j_5^\mu - \mathbf{A}\rangle, \quad (8)$$

where $\mathbf{A}$ denotes the perturbative anomaly, produced by the measure non-invariance [12]. Finally

$$\langle\partial_\mu j_5^\mu\rangle = \langle\mathbf{A}\rangle. \quad (9)$$

(R-transformation are considered similarly).

Now, let us note, that the derivation of, say, [10] essentially use the form of exact results. Therefore, we are tempted to reverse the arguments and try to obtain exact results from anomalies. Here we will consider only the most interesting case of N=1 supersymmetric theories. The detailed description of the method and some simple examples can be found in [8].

5 Effective Lagrangian for N=1 supersymmetric theories. General structure

Let us apply (9) to N=1 supersymmetric $SU(N_c)$ Yang-Mills theory with N_f matter supermultiplets. The vacuum expectation value of the perturbative anomaly (3) is given by

$$\langle \partial_\mu J^\mu_x \rangle = \Big(N_f - N_c - x N_f \Big) \frac{1}{16\pi^2} \mathrm{Im} \int d^2\theta\; u. \qquad (10)$$

where $u \equiv \langle \mathrm{tr} W^2 \rangle$. Therefore, the effective Lagrangian should depend in particular on $S = \mathrm{tr} W^2$. This result is not new. At the perturbative level the similar investigation was made in [9]. However, in this paper we do not intend to restrict ourselves by the frames of perturbation theory. Therefore, we can not assume, that $u = \mathrm{tr} W^2$.

Comparing (10) with (6) and taking into account, that the equality should be satisfied for all x, we obtain

$$\begin{aligned} &2w - 2\frac{\partial w}{\partial S} S = -\frac{1}{16\pi^2}(N_f - N_c) u; \\ &\frac{\partial w}{\partial v} v = -\frac{1}{16\pi^2} N_f u. \end{aligned} \qquad (11)$$

This equation is very similar to the results of [9]. Nevertheless, there is a crucial difference: $u \neq S$. Therefore, one can only conclude that

$$2w - 2\frac{\partial w}{\partial S} S - \frac{N_f - N_c}{N_f} \frac{\partial w}{\partial v} = 0. \qquad (12)$$

This equation corresponds to the exact conservation of R-symmetry at nonperturbative level. The similar condition was used in [6, 7], although the dependence $w = w(S)$ was ignored. Of course, it is quite clear, that integrating out S yields ADS superpotential [13] and corresponds to imposing the condition

$$\frac{\partial w}{\partial S} = 0. \qquad (13)$$

Nevertheless, the presence of S-field is a strict consequence of (11) and integrating it out should be propely substantiated (if it is possible).

The solution of (12) should agree with instanton calculations. It is easily verified, that the only solution of (12), agreeing with (7)[1], is

$$w = \frac{1}{32\pi^2} S f(z); \qquad f(z) = f_{pert}(z) + \sum_{n=1}^{\infty} c_n z^n, \tag{14}$$

where

$$z \equiv \frac{\Lambda^{3N_c - N_f}}{v^{2N_f} S^{N_c - N_f}} \tag{15}$$

is a dimensionless parameter.

In the final result z should be written in terms of gauge invariant variables. The derivation, made in the [8], gives

$$\begin{aligned} z &= \frac{\Lambda^{3N_c - N_f}}{\det M \; S^{N_c - N_f}}, \qquad N_f < N_c; \\ z &= \frac{\Lambda^{3N_c - N_f} S^{N_f - N_c}}{\det M - (\tilde{B}^{A_1 A_2 \ldots A_{N_f - N_c}} M_{A_1}^{B_1} M_{A_2}^{B_2} \ldots M_{A_{N_f - N_c}}^{B_{N_f - N_c}} B_{B_1 B_2 \ldots B_{N_f - N_c}})}, \\ & \qquad N_f \geq N_c. \end{aligned} \tag{16}$$

In order to define f_{pert} we note, that at the perturbative level $u = S$. Therefore, in this case (11) gives

$$\frac{\partial f_{pert}}{\partial z} z = 1, \tag{17}$$

so that

$$w = \frac{1}{32\pi^2} S \Big(\ln z + \sum_{n=1}^{\infty} c_n z^n \Big). \tag{18}$$

Substituting it to (11), we obtain

$$u = S \Big(1 + \sum_{n=1}^{\infty} n c_n z^n \Big), \tag{19}$$

[1] It corresponds to $g_n(x) = \frac{1}{32\pi^2} c_n x^{n(N_c - N_f)}, \quad n \geq 1$ in (7).

that defines all anomalies in the theory according to (10). At the perturbative level both (18) and (19) are certainly in agreement with [9].

6 Effective Lagrangian for N=1 supersymmetric theories. Exact result

Let us try to define f exactly. The general structure of the holomorphic superpotential, found in section 5, is similar to the structure of the nonperturbative prepotential in the N=2 supersymmetric Yang-Mills theory [14]. In the latter case the relation between perturbative and nonperturbative anomalies leads to Picard-Fuchs equations [8], that can be used for derivation of exact results. Is it possible to extend this approach to the case of N=1 supersymmetry?

First we substitute (14) into (11), that gives

$$S\frac{df}{dz}z = u \tag{20}$$

(and therefore u/S depends only on z).

The way to solve this equation is indicated by the analogy with N=2 supersymmetric $SU(2)$ Yang-Mills theory. In terms of N=1 superfields its action is written as

$$\frac{1}{16\pi}\mathrm{Im}\int d^4xd^2\theta\left(\frac{d^2F}{d\phi^2}W^2+\frac{1}{2}\int d^2\bar{\theta}\frac{dF}{d\phi}\phi^+\right). \tag{21}$$

Let us compare it with

$$\Gamma_a = \frac{1}{32\pi^2}\mathrm{Re}\int d^4xd^2\theta\ Sf(z) \tag{22}$$

and introduce $a \equiv z^{-1/4}$ (this choice of the power will be explained below). The first term in (21) will coincide with (22) if

$$-2\pi i\frac{d^2F}{da^2} \equiv f; \qquad \frac{d^2U}{da^2} \equiv \frac{u}{S}. \tag{23}$$

Then (20) takes the form

$$F + F_D = \frac{2i}{\pi} U, \tag{24}$$

where

$$F_D = F - a a_D; \qquad a_D = \frac{dF}{da}. \tag{25}$$

This equation immediately leads to Picard-Fuchs equation [8]

$$\left(\frac{d^2}{da^2} + L(U) \right) \begin{pmatrix} a \\ a_D \end{pmatrix} = 0, \tag{26}$$

where $L(U)$ is a function depending on the structure of singularities in the U-plane. At the perturbative level

$$L(U) = \frac{1}{4U^2} \tag{27}$$

but the perturbative solution does not satisfy the requirement [3, 15]

$$\mathrm{Im}\, \tau > 0, \qquad \text{where} \qquad \tau = \frac{d^2 F}{da^2} = \frac{da_D}{da} = \frac{i}{2\pi} f, \tag{28}$$

that is derived exactly as in the N=2 case. Therefore, two singularities (at $U = 0$ and $U = \infty$) are impossible.

To find the structure of singularities let us note, that the solution (18) should contain all positive powers of z and, therefore, is invariant under Z_4 transformations $a \to e^{i\pi k/2} a$. Taking into account (23) and (19) we conclude, that the corresponding transformations in the U-plane are $U \to e^{i\pi k} U$. Thus, the singularities should come in pairs: for each singularity at $U = U_0$ there is another one at $U = -U_0$.

Therefore, the considered model is completely equivalent to N=2 supersymmetric SU(2) Yang-Mills theory without matter and the only possible form of Picard-Fuchs equation (up to the redefinition of Λ) is

$$\left(\frac{d^2}{dU^2} + \frac{1}{4(U^2 - 1)} \right) \begin{pmatrix} a \\ a_D \end{pmatrix} = 0 \tag{29}$$

with the solution [3, 15]

$$a(U) = \frac{\sqrt{2}}{\pi}\int\limits_{-1}^{1} dx \frac{\sqrt{x-U}}{\sqrt{x^2-1}};$$

$$a_D(U) = \frac{\sqrt{2}}{\pi}\int\limits_{1}^{U} dx \frac{\sqrt{x-U}}{\sqrt{x^2-1}}. \qquad (30)$$

Its uniqueness and, therefore, the uniqueness of the choice (29) was proven in [16].

The function F can be found by

$$\frac{dF}{dU} = a_D \frac{da}{dU}. \qquad (31)$$

Its general structure is well known to be

$$F = -\frac{i}{\pi} a^2 \Big(\ln a + \sum_{n=0}^{\infty} F_n a^{-4n} \Big), \qquad (32)$$

so that

$$f = -2\pi i \frac{d^2 F}{da^2} = -4 \ln a + \sum_{n=0}^{\infty} f_n a^{-4n} = \ln z + \sum_{n=0}^{\infty} f_n z^n. \qquad (33)$$

And now it is quite clear, that the choice $a = z^{-1/4}$ was made to obtain the true structure of instanton corrections (14).

7 Conclusion.

We present a new method for obtaining exact results using the relation between perturbative and exact anomalies as a starting point. Its application to N=1 SUSY theories allowed to solve the long standed problem of agreement between instanton calculations and exact results. (The disagreement was present for $N_c > N_f - 1$). The proposed approach authomatically leads to the presence of gluino condensate, that was ignored in all papers devoted both to exact results and instanton calculations in N=1 SUSY theories. It is this dependence, that gives the required agreement. It is proved by the investigation of the collective coordinate measure tranformation law.

References

[1] G.t'Hooft, *Phys.Rev.Lett.* **37**, (1976), 8;
G.t'Hooft, *Phys.Rev.* **D 14**, (1976), 3432.

[2] M.Shifman, A.Vainstein, V.Zakharov, *Nucl.Phys.* **B 165**, (1980), 45;
V.Novikov, M.Shifman, A.Vainstein, V.Zakharov, *Nucl.Phys.* **B 229**, (1983), 407;
V.Novikov, M.Shifman, A.Vainstein, V.Zakharov, *Nucl.Phys.* **B 260**, (1985), 157.

[3] N.Seiberg and E.Witten, *Nucl.Phys.* **B 426**, 19, (1994).

[4] D.Finnell, P.Pouliot, *Nucl.Phys.* **B 453**, 225, (1995).

[5] A.Yung, *Nucl.Phys.* **B 485**, 38, (1997).

[6] I.Affleck, M.Dine, N.Seiberg, *Nucl.Phys.* **B 241**, (1984), 493.

[7] N.Seiberg, *Phys.Rev.* **D 49**, (1994), 6857.

[8] P.Pronin and K.Stepanyantz, *Anomalies and nonperturbative results*, hep-th/9803185.

[9] G.Veneziano, S.Yankelowicz, *Phys.Let.* **B 113**, (1982), 231;
T.Taylor, G.Veneziano, S.Yankelowicz, *Nucl.Phys.* **B 218**, (1983), 493.

[10] M.Matone, *Phys.Lett.* **B 357**, (1995), 342.

[11] P.Howe, P.West, *Nucl.Phys.* **B 486**, (1997), 425.

[12] K.Fujikava, *Phys.Rev.Lett.* **42**, 1195, (1979);
K.Fujikava, *Phys.Rev.Lett.* **44**, 1733, (1980).

[13] K.Intriligator, R.Leith, N.Seiberg, *Phys.Rev.* **D 50**, (1994), 1092.

[14] N.Seiberg, *Phys.Lett.* **B 206**, 75, (1988).

[15] A.Bilal, *Duality in N=2 SUSY SU(2) Yang-Mills theory: A pedagogical introduction to the work of Seiberg and Witten*, hep-th/ 9601007.

[16] G. Bonelli, M.Matone, M. Tonin, *Phys. Rev.* **D 55**, (1997), 6466.