Field Theoretic Issues in Low Energy Effective Field Theory

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Abstract

It is shown that low energy effective field theory gives everything relevant at low energies, not only S-matrix elements but also off-shell correlation functions. The O(N) linear and non-linear sigma models are discussed as an example.

1 Introduction

Low energy effective field theory was first introduced as a convenient tool to reproduce the results of current algebra [1]: it automatically gives S-matrix elements which are unitary and possess the right symmetry. The idea was generalized to theories with an arbitrary global continuous symmetry group G which breaks down spontaneously to a subgroup H.[2] At low energies only the massless Nambu-Goldstone bosons can be produced. Their fields take values in the coset space G/H, and the lagrangian is invariant under G. In principle the lagrangian has an infinite number of parameters, but only a couple of them, so-called the least irrelevant parameters, are important phenomenologically.

Usually an effective field theory is used to find the most general form of the S-matrix elements at low energies. For example, if one is interested in the dynamics of pions in QCD, one can use the chiral lagrangian to derive the S-matrix elements with a few parameters [1, 3]. With the chiral lagrangian, one can reproduce all the results of current algebra and PCAC. A simpler example is the O(N) linear sigma model in the broken phase in which the

O(N) is spontaneously broken to O(N-1), producing $N-1$ massless Nambu-Goldstone bosons. At low energies the physics can be described by the O(N) non-linear sigma model.

In this talk I have discussed mainly two issues. One is the off-shell renormalization properties of the O(N) non-linear sigma model (section 3). The other is how to use the O(N) non-linear sigma model to reproduce the off-shell correlation functions of the O(N) linear sigma model (section 4). It is not merely the S-matrix elements that the low energy effective field theory can give: it gives everything, including off-shell correlation functions, up to phenomenological parameters.

Most results in this talk have already appeared in ref. [4].

2 Renormalization group analysis

To understand the relation between the O(N) linear and non-linear sigma models, it is important to understand the renormalization group (RG) equations in each model. Consider the linear sigma model first. The model has an N-component field ϕ^I $(I = 1, ..., N)$ whose expectation value is oriented in the N-th direction: $\left\langle \phi^I \right\rangle \propto \delta^{I,N}$. The lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^I \partial_\mu \phi^I + \frac{\lambda}{8}\left(\phi^I\phi^I - \frac{m^2}{\lambda}\right)^2 . \tag{1}$$

The parameters satisfy the following RG equations:

$$\frac{d\lambda}{dt} = \beta(\lambda) = -\frac{N+8}{(4\pi)^2}\,\lambda^2 + \mathrm{O}\left(\lambda^3\right) \tag{2}$$

$$\frac{dm^2}{dt} = (2+\beta_m(\lambda))m^2 = \left(2 - \frac{N+2}{(4\pi)^2}\,\lambda + \mathrm{O}(\lambda^2)\right) m^2 \tag{3}$$

$$\frac{d\phi^I}{dt} = (1+\gamma(\lambda))\phi^I = (1+\mathrm{O}(\lambda^2))\phi^I . \tag{4}$$

Next consider the non-linear sigma model. The Goldstone boson field is given by an N-component unit vector Φ^I satisfying

$$\Phi^I \Phi^I = 1 . \tag{5}$$

Again the expectation value is taken in the N-th direction: $\left\langle \Phi^I \right\rangle \propto \delta^{I,N}$. The lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{eff} = \frac{1}{g_{-2}^2} \Bigg[& g_{-2} \frac{1}{2} \partial_\mu \Phi^I \partial_\mu \Phi^I \\
& + g_{-4}^{(1)} \left(\partial_\mu \Phi^I \partial_\mu \Phi^I \right)^2 + g_{-4}^{(2)} \partial_\mu \Phi^I \partial_\nu \Phi^I \partial_\mu \Phi^J \partial_\nu \Phi^J + g_{-4}^{(3)} \partial^2 \Phi^I \partial^2 \Phi^I \\
& + \quad \text{terms with more derivatives} \quad \Bigg] .
\end{aligned} \tag{6}$$

Here, the parameter g_{-2} is of order $\frac{1}{v^2} \equiv \frac{\lambda}{m^2}$, and $g_{-4}^{(i)}$'s are of order g_{-2}^2. The terms neglected involve parameters with more negative dimensions: the terms with $2n$ derivatives are multiplied by parameters g_{-2n} with $-2n$ dimensions which are of order g_{-2}^n.

The parameters satisfy the RG equations of the following form:

$$\frac{d}{dt}\, g_{-2} = -2 g_{-2} \tag{7}$$

$$\frac{d}{dt}\, g_{-4}^{(i)} = -4 g_{-4}^{(i)} - z^{(i)} g_{-2}^2 \, . \tag{8}$$

The RG equation of g_{-2n} involves the parameters with less negative dimensions. This means that more and more negative dimensional parameters are generated by renormalization. Simply said, the model is non-renormalizable.

The parameters of the low energy effective theory must be given in terms of λ and m^2 of the linear sigma model. Let

$$g_{-2} = \frac{f(\lambda, \ln m^2)}{m^2} \tag{9}$$

$$g_{-4}^{(i)} = \frac{h^{(i)}(\lambda, \ln m^2)}{m^4} \tag{10}$$

The functions f and h's are constrained by the RG equations. Eqns. (3, 7) imply that

$$\frac{d}{dt}\, f = \beta_m f \, . \tag{11}$$

Solving this, one obtains

$$f = (c_0 + c_1 \lambda_{ph} + c_2 \lambda_{ph}^2 + \ldots) \exp\left[\frac{\ln m^2}{2 + \beta_m} (\beta_m - \beta \frac{d}{d\lambda}) \right] \lambda \, , \tag{12}$$

where

$$\lambda_{ph} \equiv \exp\left[-\frac{\ln m^2}{2+\beta_m}\beta\frac{d}{d\lambda}\right]\lambda \tag{13}$$

is a RG invariant, and $c_0, c_1, c_2, \ldots$ are constants. The constants are phenomenological parameters, and it is the role of the linear sigma model to determine them. Similarly, one can determine the function $h^{(i)}$ up to a RG invariant function $d_0^{(i)} + d_1^{(i)}\lambda_{ph} + d_2^{(i)}\lambda_{ph}^2 + \ldots$.

3 Off-shell renormalization of the O(N) non-linear sigma model

As in any renormalization of a field theory, both the parameters and the fields must be renormalized in the O(N) non-linear sigma model. The model is non-renormalizable in the sense that at each loop order new parameters (with more negative dimensions) must be introduced. Otherwise the renormalization program works just as well as for any renormalizable theories except for one point: not only the single field Φ^I must be renormalized but also its products must be renormalized.[1] The dimension regularization is used with $D \equiv 4-\epsilon$ as the dimensionality of space. The $\overline{\mathrm{MS}}$ scheme is used with the renormalization scale $\bar{\mu} = 1$.

The bare parameter $G_{-4}^{(i)}$ is given in terms of the renormalized parameter $g_{-4}^{(i)}$ as

$$G_{-4}^{(i)} = \frac{z^{(i)}}{\epsilon} g_{-2}^2 + g_{-4}^{(i)} , \tag{14}$$

where $z^{(i)}$ is a constant. The renormalized parameter satisfies the RG equation (8). Next, the field Φ^I must be renormalized:

$$[\Phi^I] = \Phi^I + \frac{g_{-2}}{\epsilon}\left(a_1(\partial_\mu\Phi^J\partial_\mu\Phi^J)\Phi^I + a_2\partial^2\Phi^I\right) + \ldots \tag{15}$$

where higher dimensional fields which are O(N) vectors are suppressed. This implies the RG equation

$$\frac{d}{dt}\,[\Phi^I] = g_{-2}\left(a_1(\partial_\mu\Phi^J\partial_\mu\Phi^J)\Phi^I + a_2\partial^2\Phi^I\right) + \ldots \tag{16}$$

[1] The necessity of renormalizing products of fields is familiar to those who have studied renormalization of composite fields in renormalizable theories. See, for example, [5].

Hence, the effective field ϕ^I_{eff} which corresponds to ϕ^I is a linear combination of $[\Phi^I]$, $\partial_\mu \Phi^J \partial_\mu \Phi^J \cdot \Phi^I$, and $\partial^2 \Phi^I$.

The product of two Φ's receives additional renormalization:

$$
\begin{aligned}
[\Phi^I(r_1)\Phi^J(r_2)] &= [\Phi^I(r_1)][\Phi^J(r_2)] \\
&+ \frac{g^2_{-2}}{\epsilon}\left(b_1\delta^{IJ} + b_2(\Phi^I\Phi^J - \frac{\delta^{IJ}}{N}) + ...\right)\delta^{(D)}(r_1 - r_2) \ . \qquad (17)
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{d}{dt}\ [\Phi^I(r_1)\Phi^J(r_2)] &= (\frac{d}{dt}\ [\Phi^I(r_1)])[\Phi^J(r_2)] + [\Phi^I(r_1)](\frac{d}{dt}\ [\Phi^J(r_2)]) \\
+ \ \ g^2_{-2}&\left(b_1\delta^{IJ} + b_2(\Phi^I\Phi^J - \frac{\delta^{IJ}}{N}) + ...\right)\delta^{(D)}(r_1 - r_2) \ . \qquad (18)
\end{aligned}
$$

Therefore, in multi-point correlation functions the field ϕ^I of the linear sigma field does not correspond to a single field ϕ^I_{eff} in the non-linear sigma model. This will be discussed further in the next section.

So far, seven renormalization constants, $z^{(1,2,3)}$, $a_{1,2}$, and $b_{1,2}$, have been introduced. No further renormalization is required at one-loop level. It turns out, though, that these constants are not all independent due to equations of motion. One finds that the non-linear sigma model is invariant under the following two transformations:

$$
\Delta z^{(1)} = -\Delta z^{(3)} = \Delta a_1 = \Delta a_2 \qquad (19)
$$

$$
\Delta a_1 = \Delta a_2 = \frac{1}{2\left(1 - \frac{1}{N}\right)}\Delta b_1 = -\frac{1}{2}\Delta b_2 \ . \qquad (20)
$$

The above invariance can be used to introduce a convention

$$
z^{(3)} = b_2 = 0 \ . \qquad (21)
$$

With this convention, explicit one-loop calculations give the following results

$$
\begin{aligned}
z^{(1)} &= \frac{1}{(4\pi)^2}\left(\frac{1}{3} + \frac{1}{2}(N-4)\right) \ , \quad z^{(2)} = \frac{1}{(4\pi)^2}\frac{2}{3} \qquad (22) \\
a_1 &= \frac{1}{(4\pi)^2}\left(\frac{1}{2} + \frac{1}{2}(N-4)\right) \ , \quad a_2 = \frac{1}{(4\pi)^2}\left(-\frac{3}{2} - \frac{1}{2}(N-4)\right) \qquad (23) \\
b_1 &= 2a_2 \ . \qquad (24)
\end{aligned}
$$

4 Off-shell equivalence

In section 2 it was shown that the phenomenological parameters g_{-2} and $g_{-4}^{(i)}$'s can be fixed by the RG up to RG invariants $c_0 + c_1\lambda_{ph} + \ldots$ and $d_0^{(i)} + d_1^{(i)}\lambda_{ph} + \ldots$. Similarly, the phenomenological fields which correspond to ϕ^I and its products can be determined up to RG invariants. To determine these RG invariants, one must compute the off-shell correlation functions of the linear sigma model explicitly.

By rearranging the one-loop calculations of the two- and four-point functions of ϕ's in the linear sigma model, one finds

$$g_{-2} \simeq \frac{\lambda}{m^2}\left(1 + \frac{\lambda}{(4\pi)^2}\left(3\ln m^2 - \frac{7}{2}\right)\right) \tag{25}$$

$$g_{-4}^{(1)} \simeq \frac{\lambda}{m^4}\left(-\frac{1}{2} + \frac{\lambda}{(4\pi)^2}\left(-\frac{1}{6}\ln m^2 + \frac{41}{36}\right)\right) \tag{26}$$

$$g_{-4}^{(2)} \simeq \frac{\lambda}{m^4}\frac{\lambda}{(4\pi)^2}\left(-\frac{1}{3}\ln m^2 + \frac{17}{18}\right) \tag{27}$$

$$g_{-4}^{(3)} \equiv 0 \tag{28}$$

The last equation is a convention. As was seen in section 2, the coefficients of the logarithms are determined by the RG, and only the remaining constants carry the essential information left undetermined by the RG.

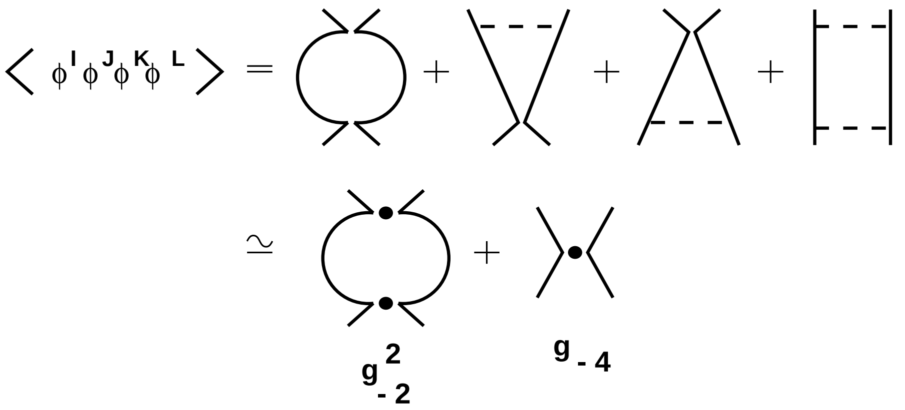

Figure: Low energy approximation of a four-point function

One also finds the following correspondence of the single fields:

$$\phi^I \quad \leftrightarrow \quad [\phi^I_{eff}] \simeq \frac{z(\lambda, m^2)}{\sqrt{g_{-2}}} \Big([\Phi^I] + g_{-2} A_1(\lambda, m^2) \partial_\mu \Phi^J \partial_\mu \Phi^J \cdot \Phi^I + g_{-2} A_2(\lambda, m^2) \partial^2 \Phi^I \Big) \tag{29}$$

where

$$g_{-2}^2 A_1 \quad \simeq \quad \frac{\lambda}{m^4} \left[-\frac{1}{2} + \frac{\lambda}{(4\pi)^2} \left(-\frac{1}{4} \ln m^2 + \frac{17}{36} \right) \right] \tag{30}$$

$$g_{-2}^2 A_2 \quad \simeq \quad \frac{\lambda}{(4\pi)^2} \left[\frac{1}{2} + \frac{\lambda}{(4\pi)^2} \left(\frac{3}{4} \ln m^2 - \frac{127}{36} \right) \right] \tag{31}$$

$$z \quad \simeq \quad 1 - \frac{\lambda}{(4\pi)^2} \frac{1}{4} \; . \tag{32}$$

The product of two fields has the following correspondence:

$$\phi^I(r_1)\phi^J(r_2) \leftrightarrow [\phi_{eff}(r_1)\phi_{eff}(r_2)] \simeq [\phi_{eff}(r_1)][\phi_{eff}(r_2)] \tag{33}$$

$$+ g_{-2} \left(B_1 \delta^{IJ} + B_2 \left(\Phi^I \Phi^J(r_1) - \frac{\delta^{IJ}}{N} \right) \right) \delta^{(D)}(r_1 - r_2) \tag{34}$$

where

$$g_{-2}^2 B_1 \quad \simeq \quad \frac{\lambda}{m^4} \left[1 + \frac{\lambda}{(4\pi)^2} \left(\frac{3}{2} \ln m^2 - \frac{62}{9} \right) \right] \tag{35}$$

$$g_{-2}^2 B_2 \quad \equiv \quad 0 \tag{36}$$

The second equation is a convention. The coefficients of the logarithms in $A_{1,2}$ and B_1 are determined by the RG. The remaining constants are the non-trivial information provided by the linear sigma model.

At one-loop level one can reproduce any off-shell correlation functions of ϕ's using the non-linear sigma model and the above rules of correspondence. If one had only the non-linear sigma model, the RG invariant constants in the parameters g_{-2}, $g_{-4}^{(i)}$'s (eqns. [25 – 27]) and the coefficients z, A_i's, B_i's (eqns. [30 – 32] and [35]) are left undetermined as phenomenological constants.

5 Conclusion

In this talk I have explained the true meaning of low energy effective field theory: it reproduces not only S-matrix elements but also off-shell correlation functions. Only the O(N) linear and non-linear sigma models have been discussed here in details. Similar analyses can be extended to QCD with quarks (chiral lagrangian) and non-abelian gauge theories with Higgs fields (electroweak theory with no Higgs).

References

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