Seiberg-Witten Systems and Whitham Hierarchies: a Short Review

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Abstract

We briefly review the Whitham hierarchies and their applications to integrable systems of the Seiberg-Witten type. The simplest example of the $N = 2$ supersymmetric $SU(2)$ pure gauge theory is considered in detail and the corresponding Whitham solutions are found explicitely.

1 Introduction

Since constructing by N.Seiberg and E.Witten the low-energy exact solution to the $N = 2$ supersymmetric $SU(2)$ pure gauge theory [1], its intensive studies have been performed from many different points of view. In particular, it was realized that this solution as well as analogous solutions in many other similar theories ($N = 2$ supersymmetric gauge theories in $4d$, $5d$ and $6d$ with different gauge groups and different matter contents) are effectively expressed in terms of integrable systems [2, 3, 4, 5, 6]. These underlying integrable systems encode symmetry properties of the solutions.

Moreover, how it was clear from the very beginning [2], the Seiberg-Witten solutions have something to do with *two* different structures: with finite-gap solutions of integrable systems of the Hitchin type or spin chains (so

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that the integrals of motion are fixed and related to the vacuum expectation values (vev's) of the scalar fields in the supersymmetric theories) *and* with the corresponding *Whitham* hierarchy constructed in the vicinity of these solutions and describing the evolution of integrals of motion. Although the role of both these structures is not quite understood yet, one can indicate associated key quantities that play the crucial role for the Seiberg-Witten construction.

The first object is prepotential which defines the low-energy effective action of the vector multiplets in $\mathcal{N} = 2$ supersymmetric theories. The notion of prepotential can be easily introduced for the finite-gap solutions: it is a function of integrals of motion (or of moduli of the complex curve describing the solution) whose second derivatives give the period matrix. As for the Whitham hierarchy, the prepotential plays there the role of the (quasiclassical) τ-function. However, it turns out that one can introduce even more – the prepotential which depends on both the Whitham times and the integrals of motion as on *independent* variables (see, e.g. [7]). Such prepotential incorporates properties of the both systems and encodes non-trivial quantities (correlators) in the physical (supersymmetric) theories (see, e.g. [8], where these correlators have been treated within the framework of the Donaldson topological theories).

In this note, we are going to review the properties of prepotential, and corresponding integrable structures arising in the context of the Seiberg-Witten solution with main emphasis on the Whitham type hierarchies. As our main illustrative example, we shall consider the supersymmetric $SU(2)$ pure gauge theory.

First of all, we consider the finite-gap solution to the integrable hierarchy underlying the Seiberg-Witten system (which is the periodic Toda chain [2]). This solution is described by complex (spectral) curve whose moduli are the integrals of motion. The symplectic structure can be defined via meromorphic differential on complex curve and the periods of this differential define the prepotential.

Then, we introduce evolution of moduli satisfying the equations of the Whitham hierarchy. After this, we find a generic solution to the Whitham hierarchy associated with the Seiberg-Witten anzatz and define the prepotential that depends on both Whitham times *and* moduli of the curve as on independent variables. We illustrate our consideration in the simplest toric example.

2 Seiberg-Witten integrable system

We are going to discuss here the simplest Seiberg-Witten solution describing the low-energy limit of the pure gauge theory with $SU(N)$ gauge group. The underlying integrable is the Toda chain of length N with periodic boundary conditions [2]. In this case, the spectral curve has the form

$$w + \frac{\Lambda^{2N}}{w} = P_N(\lambda) \tag{1}$$

or, in hyperelliptic parametrization,

$$y^2 = P_N(\lambda)^2 - 4\Lambda^{2N} \tag{2}$$

where $P_N(\lambda) \equiv \lambda^N - \sum_{k=2}^N u_k \lambda^{N-k}$ is a polynomial of degree N so that the genus of the curve is $N-1$, Λ is a scale parameter of the physical theory, which can be easily put equal to unity, $y = w - \frac{\Lambda^{2N}}{w}$. All the dependence on moduli (vev's of the scalars in the physical theory) is hidden in coefficients u_k's of $P_N(\lambda)$. Note that a finite-gap solution to integrable system usually corresponds to a spectral curve with *marked points*. In the particular case of Toda chain, these are two points $w = 0$ and $w = \infty$ (and $\lambda = \infty_\pm$ on the two λ-sheets).

To define the prepotential, one also needs a generating differential that can be constructed using the symplectic form Ω of integrable system. The result for the Toda chain differential is

$$dS_{SW} = \text{``}\delta^{-1}\Omega\text{''} = \lambda dz \tag{3}$$

the precise meaning of "$\delta^{-1}\Omega$" being explained along with the origin of the whole formula, e.g. in [9]. Here

$$dz \equiv d\log w = \frac{dw}{w} = \frac{dP_N}{y} \tag{4}$$

The important property of the, generally meromorphic (in the marked points), differential dS_{SW} is that its derivatives w.r.t. moduli are *holomorphic* differentials. Now the prepotential can be defined by

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i} \tag{5}$$

where[1]

$$a_i = \oint_{A_i} dS_{SW} \qquad a_i^D = \oint_{B_i} dS_{SW} \tag{6}$$

with $A_i \circ B_j = \delta_{ij}$ being canonically normalized cycles on the complex curve. Note that

$$\frac{\partial dS_{SW}}{\partial a_i} = d\omega_i, \qquad \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = \frac{\partial a_i^D}{\partial a_j} = \mathcal{T}_{ij} \tag{7}$$

where $d\omega_i$ are canonical holomorphic differentials and $\mathcal{T}_{ij}$ is the period matrix of the complex curve.

$SU(2)$ **example.** In the simplest $SU(2)$ case, $P_2(\lambda) = \lambda^2 - u$, the curve (1) turns into

$$2\Lambda^2 \cosh z = \lambda^2 - u \tag{8}$$

and coincides with the equation relating constant Hamiltonian (energy) with the co-ordinate $z = iq$ ($z = q$) and momentum $\lambda = p$ of a particle moving into the "1-dimensional" sine-Gordon (sinh-Gordon) – or 2-site Toda-chain potential:

$$2\Lambda^2 \cos q = p^2 - u \qquad \left(2\Lambda^2 \cosh q = p^2 - u\right) \tag{9}$$

The Toda system as any 1-dimensional system with conserved energy is integrable (since one only needs a single integral of motion). One can simply perform the integration of (9)

$$dt = \frac{dq}{p} = \frac{dz}{\lambda} = 2\frac{d\lambda}{y} \tag{10}$$

[1]We define the symbols $\oint$ and res with additional factors $(2\pi i)^{-1}$ so that

$$\mathrm{res}_0 \frac{d\xi}{\xi} = -\mathrm{res}_\infty \frac{d\xi}{\xi} = \oint \frac{d\xi}{\xi} = 1$$

This explains the appearance of $2\pi i$ factors in the Riemann identities (like eq.(54) below) and thus in all definitions of sect.5. Accordingly, the theta-functions are periodic with period $2\pi i$, and

$$\frac{\partial \theta(\vec{\xi}|\mathcal{T})}{\partial \mathcal{T}_{ij}} = i\pi \partial_{ij}^2 \theta(\vec{\xi}|\mathcal{T})$$

since periods of the Jacobi transformation $\xi_i \equiv \int^\xi d\omega_i$ belong to $2\pi i\,(\mathbb{Z} + \mathcal{T}\mathbb{Z})$.

3 Generalities on Whitham

Now we turn to general description of the Whitham hierarchy. Following I.Krichever [11, 12], we consider initially a *local* system of holomorphic functions Ω_I on complex curve, i.e. functions of some local parameter ξ in a neighborhood of a point P. One can introduce a set of parameters t_I and define on the space (ξ, t_I) a 1-form

$$\omega = \sum_I \Omega_I(P, \mathbf{t})\delta t_I \tag{11}$$

Considering its total external derivative $\delta\omega = \sum_I \delta\Omega_I(P, \mathbf{t}) \wedge \delta t_I$, where $\delta\omega = \partial_\xi \Omega_I \delta\xi \wedge \delta t_I + \partial_J \Omega_I \delta t_I \wedge \delta t_J$, one can define general *Whitham equations* as

$$\delta\omega \wedge \delta\omega = 0 \tag{12}$$

so that it is necessary to check the independent vanishing of two different terms: δt^4 and $\delta t^3 \delta\xi$. The second term gives

$$\sum \partial_\xi \Omega_{[I} \partial_J \Omega_{C]} = 0 \tag{13}$$

where [...] means antisymmetrization. Fixing now some $I = I_0$ and introducing the "Darboux" co-ordinates

$$t_{I_0} \equiv x \qquad \Omega_{I_0}(P, \mathbf{t}) \equiv p \tag{14}$$

one can get from (13) the Whitham equations in their standard form [12]

$$\begin{gathered} \partial_I \Omega_J - \partial_J \Omega_I + \{\Omega_I, \Omega_J\} = 0 \\ \{\Omega_I, \Omega_J\} \equiv \frac{\partial \Omega_I}{\partial x}\frac{\partial \Omega_J}{\partial p} - \frac{\partial \Omega_J}{\partial x}\frac{\partial \Omega_I}{\partial p} \end{gathered} \tag{15}$$

The explicit form of the equations (15) strongly depends on choice of the local co-ordinate p. Equations (12), (13) and (15) are defined only locally and have a huge amount of solutions.

Now let us note that the Whitham equations (15) can be considered as compatibility conditions of the system

$$\frac{\partial \lambda}{\partial t_I} = \{\lambda, \Omega_I\} \tag{16}$$

The function λ can be used itself in order to define a new local parameter. In this co-ordinate, the system (15) turns into [12]

$$\left.\frac{\partial\Omega_I(\lambda,t)}{\partial t_J}\right|_{\lambda=const} = \left.\frac{\partial\Omega_J(\lambda,t)}{\partial t_I}\right|_{\lambda=const} \tag{17}$$

or for the differentials $d\Omega_I \equiv d_\Sigma\Omega_I = \frac{\partial\Omega_I}{\partial\lambda}d\lambda$

$$\left.\frac{\partial d\Omega_I(\lambda,t)}{\partial t_J}\right|_{\lambda=const} = \left.\frac{\partial d\Omega_J(\lambda,t)}{\partial t_I}\right|_{\lambda=const} \tag{18}$$

The Whitham equations in the form (17) imply the existence of potential S such that

$$\Omega_I(\lambda,t)|_{\lambda=const} = \left.\frac{\partial S(\lambda,t)}{\partial t_I}\right|_{\lambda=const} \tag{19}$$

With this potential, the 1-form ω (11) can be rewritten as

$$\omega = \delta S - \frac{\partial S(\lambda,t)}{\partial\lambda}d\lambda \tag{20}$$

To fill all these formulas by some real content, one can consider interesting and important examples of solutions to the Whitham hierarchy arising when one takes the basis functions Ω_I to be the "quasi-energies" of the finite-gap solution; these are the (globally multivalued) functions whose periods determine the "phase" of the quasi-periodic Baker-Akhiezer function corresponding to this finite-gap solution. The potential should be then identified with

$$\begin{gathered} S = \int^P dS \\ \frac{\partial S}{\partial t_I} = \int^P \frac{\partial dS}{\partial t_I} = \int^P d\Omega_I = \Omega_I \end{gathered} \tag{21}$$

where dS is a generating differential for the finite gap system [9]. Usually for KP/Toda hierarchy these functions are taken to be "half"-multivalued, i.e. their **A**-periods are fixed to be zero: $\oint_{\mathbf{A}} d\Omega_I = 0$. The equations (21) are obviously satisfied in a trivial way in the case of the (finite-gap solutions to) KP/Toda theory, when $\{t_I\}$ are taken to be *external* parameters and $\{\Omega_I\}$ do not depend themselves on $\{t_I\}$. The idea, however, is to deform this trivial solution to (12), (13) and (15) into a nontrivial one by *a flow* in the moduli space. Practically it means that the formulas (21) should be preserved even when $\{\Omega_I\}$ *depend* on $\{t_I\}$, but in a special way – determined by the equations of Whitham hierarchy. In other words, the Whitham equations correspond to the dynamics on moduli space of complex curves and the moduli become depending on the Whitham times.

To realize all this technically, one choose $d\Omega_I$ to be a set of (normalized) differentials on the complex curve holomorphic outside the marked points where they have some fixed behavior. An important example of the λ co-ordinate is the *hyperelliptic* co-ordinate on hyperelliptic curve

$$y^2 = R(\lambda) = \prod_{\beta}^{2g+2} (\lambda - r_\beta) \tag{22}$$

Properly choosing the differentials $d\Omega_I$ and expanding the equations (18) near the point $\lambda = r_\alpha$ one gets the Whitham equations over finite-gap solutions

$$\frac{\partial r_\alpha}{\partial t_I} = \left.\frac{d\Omega_J}{d\Omega_I}\right|_{\lambda = r_\alpha} \frac{\partial r_\alpha}{\partial t_J} \equiv v_{IJ}^{(\alpha)}(\mathbf{r}) \frac{\partial r_\alpha}{\partial t_J} \tag{23}$$

The set of ramification points $\{\mathbf{r}\} = \{r_\beta\}$ is usually called Riemann invariants for the Whitham equations.

The particular case of Whitham hierarchy associated with the finite-gap solution of the KdV hierarchy described by the hyperelliptic curve

$$y^2 = R(\lambda) = \prod_{\beta}^{2g+1} (\lambda - r_\beta) \tag{24}$$

was considered first in [13].

Note also that in the framework of Whitham hierarchy it becomes possible to introduce an analog of the tau-function, again a generating function for the solutions, which can be symbolically defined as [12]

$$\log \mathcal{T}_{\mathrm{Whitham}} = \int_\Sigma \bar{d}S \wedge dS \tag{25}$$

where the two-dimensional integral over Σ is actually "localized" on the non-analyticities of S.

4 Whitham hierarchy for the finite-gap solutions

Now let us discuss how one can choose the differentials $d\Omega_I$ in order to get non-trivial solutions to Whitham hierarchy. They can be defined for the following set of data:

- complex curve (Riemann surface) of genus g;
- a set of punctures P_i (marked points);
- co-ordinates ξ_i in the vicinities of the punctures P_i;
- pair of differentials, say, $(d\lambda,\ dz)$ with fixed periods.

We start with considering the simplest case of a single puncture, say at $\xi = \xi_0 = 0$. This situation is typical, e.g., for the Whitham hierarchy in the vicinity of the finite-gap solutions to KP or KdV hierarchies. For instance, in the KdV case the spectral curve has the form (24) so that the marked point is at $\lambda = \infty$ but, in contrast to the Toda curve (2), there is only one infinite point (since it is the ramification point).

Given this set of data, one can introduce meromorphic differentials with the poles only at some point P_0 such that in some local co-ordinate ξ: $\xi(P_0) = \xi_0 = 0$

$$d\Omega_n \underset{P\to P_0}{=} \left(\xi^{-n-1} + O(1)\right) d\xi, \quad n \geq 1 \tag{26}$$

This condition defines $d\Omega_n$ up to arbitrary linear combination of g holomorphic differentials $d\omega_i$, $i = 1, \ldots, g$ and there are two different natural ways to fix this ambiguity. The first way already mentioned above is to require that $d\Omega_n$ have vanishing A-periods,

$$\oint_{A_i} d\Omega_n = 0 \quad \forall i, n \tag{27}$$

Their generating functional ($\zeta \equiv \xi(P')$, $\xi \equiv \xi(P)$)

$$W(\xi, \zeta) = \sum_{n=1}^{\infty} n\zeta^{n-1} d\zeta d\Omega_n(\xi) + \ldots \tag{28}$$

is well known in the theory of Riemann surfaces. It can be expressed through the Prime form $E(P,P')$ (see Appendix A of [7] for details): [2]

$$W(P,P') = \partial_P \partial_{P'} \log E(P,P') \tag{29}$$

Such $W(P,P')$ has a second order pole on diagonal $P \to P'$,

$$W(\xi,\zeta) \sim \frac{d\xi d\zeta}{(\xi-\zeta)^2} + O(1) = \sum_{n=1}^{\infty} n \frac{d\xi}{\xi^{n+1}} \zeta^{n-1} d\zeta + O(1) \tag{30}$$

It is the differentials (26) that should be related with the potential differential dS by

$$d\Omega_n = \frac{\partial dS}{\partial t_n} \tag{31}$$

The second way to normalize differentials is to impose the condition

$$\frac{\partial d\hat{\Omega}_n}{\partial \text{ moduli}} = \text{holomorphic} \tag{32}$$

so that these differentials becomes similar to the generating differential dS_{SW} of integrable system (see sect.2).

Now let us specialize this description to the case considered in sect.2, i.e. to the family of spectral curves (1). The first problem is that naively there are no solutions to eq.(32) because the curves (1) are spectral curves of the

[2] For example, for genus $g=1$ in the co-ordinate $\xi \sim \xi+1 \sim \xi+\tau$ the formulas acquire the form

$$d\Omega_1 = (\wp(\xi) - const)d\xi,\ d\Omega_2 = -\frac{1}{2}\wp'(\xi)d\xi,\ \ldots, d\Omega_n = \frac{(-)^{n+1}}{n!}\partial^{n-1}\wp(\xi)d\xi$$

and

$$W(\xi,\xi') = \sum_{n=1}^{\infty} \frac{(-)^{n+1}\xi'^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial\xi^{n-1}}\wp(\xi)d\xi d\xi' - const\cdot d\xi d\xi' = \wp(\xi-\xi')d\xi d\xi' =$$

$$= \partial_\xi \partial_{\xi'} \log \theta_*(\xi - \xi')$$

where $*$ denotes the (on torus the only one) odd theta-characteristic. For $g=1$ $\nu_*^2(\xi) = \theta_{*,i}(0)d\omega_i$ is just $d\xi$. Let us also point out that choosen in this way co-ordinate ξ is not convenient from the point of view of Whitham hierarchy since its "periods" ($\tau = \oint_B d\xi$) depend on moduli of the curve.

Toda-chain hierarchy (not of KP/KdV type). The difference is that adequate description in the Toda case requires *two* punctures instead of one. As already mentioned in sect.2 above, the curves (1) have two marked points and there exists a function w with the N-degree pole and zero at two (marked) points $\lambda = \infty_\pm$, where $\pm$ labels two sheets of the hyperelliptic representation (1), $w(\lambda = \infty_+) = \infty$, $w(\lambda = \infty_-) = 0$. Accordingly, there are two families of the differentials $d\Omega_n$: $d\Omega_n^+$ with the poles at ∞_+ and $d\Omega_n^-$ with the poles at ∞_-.

However, there are *no* differentials $d\hat{\Omega}^\pm$, only $d\hat{\Omega}_n = d\hat{\Omega}_n^+ + d\hat{\Omega}_n^-$, i.e. condition (32) requires $d\hat{\Omega}_n$ to have the poles at both punctures. Moreover, the coefficients in front of $w^{n/N}$ at ∞_+ and $w^{-n/N}$ at ∞_- (26) coincide (in Toda-hierarchy language, this is the Toda-*chain* case with the same dependence upon negative and positive times).

The differentials $d\hat{\Omega}_n$ for the family (1) have the form [7]:

$$d\hat{\Omega}_n = R_n(\lambda)\frac{dw}{w} = P_+^{n/N}(\lambda)\frac{dw}{w} \tag{33}$$

The polynomials $R_n(\lambda)$ of degree n in λ are defined by the property that $P'\delta R_n - R_n'\delta P$ is a polynomial of degree less than $N-1$. Thus, $R_n(\lambda) = P_+^{n/N}(\lambda)$, where $\left(\sum_{k=-\infty}^{+\infty} c_k\lambda^k\right)_+ = \sum_{k=0}^{+\infty} c_k\lambda^k$. For example[3]:

$$\begin{aligned}
R_0 &= 1,\\
R_1 &= \lambda,\\
R_2 &= \lambda^2 - \frac{2}{N}u_2,\\
R_3 &= \lambda^3 - \frac{3}{N}u_2\lambda - \frac{3}{N}u_3,\\
R_4 &= \lambda^4 - \frac{4}{N}u_2\lambda^2 - \frac{4}{N}u_3\lambda - \left(\frac{4}{N}u_4 + \frac{2(N-4)}{N^2}u_2^2\right),\\
&\dots
\end{aligned} \tag{34}$$

These differentials satisfy (32) provided the moduli-derivatives are taken at constant w (not λ!). Thus, the formalism of the previous section is applicable

[3]In the case of the Toda chain, since there are two punctures, one can consider the meromorphic differential with the simple pole at each of them.

for the local parameter $\xi = w^{\mp 1/N}$.[4]

Thus, among the data one needs for the definition of solution to the Whitham hierarchy associated with the Toda chain, there are the punctures at $\lambda = \infty_\pm$ and the relevant co-ordinates in the vicinities of these punctures $\xi \equiv w^{-1/N} \sim \lambda^{-1}$ at ∞_+ and $\xi \equiv w^{+1/N} \sim \lambda^{-1}$ at ∞_-. The parametrization in terms of w, however, does not allow to use the advantages of hyperelliptic form (2).

Let us consider our simplest explicit example.

Whitham equations for the $SU(2)$ case. In the case of 2-site periodic Toda chain with the spectral curve (8), one can restrict himself to the first two differentials $d\Omega_n$ and the two first times t_0, t_1. Then, the Whitham equations

$$\frac{\partial d\Omega_i}{\partial t_j} = \frac{\partial d\Omega_j}{\partial t_i} \tag{35}$$

are reduced to the only equation

$$\frac{\partial d\Omega_0}{\partial t_1} = \frac{\partial d\Omega_1}{\partial t_0} \tag{36}$$

To write it down explicitely one should remember that

- There are two independent differentials (see (33) and (34) and formulas of sect.2)

$$\begin{aligned} d\Omega_0 &= dz + \gamma_0 \frac{d\lambda}{y} = \left(1 + \frac{\gamma_0}{2\lambda}\right) dz \\ d\Omega_1 &= \lambda dz + \gamma_1 \frac{d\lambda}{y} = \left(\lambda + \frac{\gamma_1}{2\lambda}\right) dz \end{aligned} \tag{37}$$

  and two corresponding Whitham times t_0 and t_1.

- The coefficients γ_i $(i = 0, 1)$ are fixed, as usual, by vanishing of the correspondent A-periods

$$\oint_A d\Omega_i = 0 \tag{38}$$

[4]Rule of thumb generally is to choose as the local co-ordinate for the Whitham hierarchy the parameter living on the bare spectral curve. For the Toda case, one may choose as a bare curve the w-cylinder (see, e.g. [6]), with the corresponding generating differential $dS = \lambda d \log w$.

i.e.

$$\gamma_1 = -\frac{a}{\sigma} \qquad \gamma_0 = -\frac{1}{\sigma} \tag{39}$$

where

$$\begin{gathered} a = \oint_A dS = \oint_A \lambda dz \\ \sigma = \oint_A \frac{d\lambda}{y} = \frac{\partial a}{\partial u} \end{gathered} \tag{40}$$

- The derivatives over moduli in (35) are taken at fixed values of z-variable, while the λ-variable depends on moduli and this dependence is given by (8). This rule leads to the relations

$$\begin{gathered} \frac{\partial \lambda}{\partial u} = \frac{1}{2\lambda} \\ \frac{\partial a}{\partial u} = \oint_A \frac{\partial \lambda}{\partial u} dz = \oint_A \frac{dz}{2\lambda} = \sigma \end{gathered} \tag{41}$$

Using these relations one easily gets from (35) the following Whitham equation on moduli

$$\frac{\partial u}{\partial t_1} = a(u) \frac{\partial u}{\partial t_0} \tag{42}$$

The function $a(u)$ is an elliptic integral. Its explicit expression depends on choice of the cycles. With the choice used in [7], i.e. when the **A**-cycles encircle the points $\lambda = \sqrt{u - 2\Lambda^2} \equiv r^-$ and $\lambda = \sqrt{u + 2\Lambda^2} \equiv r^+$,

$$a = \frac{2}{\pi} r^+ E(k), \qquad \sigma = \frac{1}{\pi r^+} K(k) \tag{43}$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and the second kinds respectively and the elliptic modulus is $k = \frac{2}{r^+}$.

The Whitham equation (42) has the general solution:

$$u = F\left(t_0 + a(u) t_1\right) \tag{44}$$

where $F(x)$ is an arbitrary function. This solution can be also rewritten in the form

$$\Phi(u) = t_0 + a(u) t_1 \tag{45}$$

where $\Phi(u)$ is the function inverse to $F(x)$.

5 Solution to the Whitham hierarchy and prepotential

Now we are going to extend the system of differentials $d\Omega_i$ introduced in the previous section to include the holomorphic differentials. This will allow us to construct the function that, as a function of one set of variables, is the prepotential of the Seiberg-Witten theory (as defined in sect.2) and in the other set of *independent* variables is the (logarithm of) τ-function of the associated solution to the Whitham hierarchy. On this way, we also get a generic solution to the associated Whitham equations.

Thus, we extend the set of Whitham equations by the equations involving the holomorphic differentials $d\omega_i$ and moduli α_i:

$$\frac{\partial d\Omega_n}{\partial \alpha_i} = \frac{\partial d\omega_i}{\partial t_n}, \qquad \frac{\partial d\omega_i}{\partial \alpha_j} = \frac{\partial d\omega_j}{\partial \alpha_i} \tag{46}$$

This system is solved by the differential dS that satisfies

$$\frac{\partial dS}{\partial \alpha_i} = d\omega_i, \quad \frac{\partial dS}{\partial t_n} = d\Omega_n \tag{47}$$

Then, the first equation in (47) implies that dS is to be looked for as a linear combination of the differentials $d\hat{\Omega}_n$, satisfying (32). Let us, following [5, 7] introduce a generating functional for $d\hat{\Omega}_n$ with infinitely many auxiliary parameters t_n:

$$dS = \sum_{n\geq 1} t_n d\hat{\Omega}_n = \sum_{i=1}^{g} \alpha_i d\omega_i + \sum_{n\geq 1} t_n d\Omega_n \tag{48}$$

The periods

$$\alpha_i = \oint_{A_i} dS \tag{49}$$

can be considered as particular co-ordinates on the moduli space. Note that these periods do not exactly coincide with (6), eq.(49) defines α_i as functions of h_k and t_n, or, alternatively, h_k as functions of α_i and t_n so that derivatives $\partial h_k/\partial t_n$ are non-trivial. In what follows we shall consider α_i and t_n,

$$t_n = \mathrm{res}_{\xi=0}\ \xi^n dS(\xi) \tag{50}$$

as independent variables so that partial derivatives w.r.t. α_i are taken at constant t_n and partial derivatives w.r.t. t_n are taken at constant α_i.

The differential dS (48) determines a generic form of the solution associated to the Seiberg-Witten type Whitham hierarchy. The Whitham dynamics itself for given dS can be formulated in terms of equations (47). Note that, if one restricts himself to the Whitham hierarchy with several first times (generally, for the genus g complex curve there are $g+n-1$ independent times, with n being the number of punctures), all higher times in (48) play the role of constants (parameters) of generic Whitham solution (see example of $SU(2)$ below).

Note that the Seiberg-Witten differential dS_{SW} is $dS_{SW} = d\hat{\Omega}_2$, i.e.

$$dS\big|_{t_n=\delta_{n,1}} = dS_{SW}, \quad \alpha^i\Big|_{t_n=\delta_{n,1}} = a^i, \quad \alpha_i^D\Big|_{t_n=\delta_{n,1}} = a_i^D \tag{51}$$

and α-variables are naturally associated with the Seiberg-Witten moduli, while t-variables – with the corresponding Whitham times (although α_i's can be also considered as variables of Whitham dynamics, c.f. (46)).

Now one can introduce the Whitham tau-function (25) whose logarithm is a *prepotential* $\mathcal{F}(\alpha_i, t_n) \equiv \log \mathcal{T}$ by an analog of conditions (5)-(6):

$$\frac{\partial \mathcal{F}}{\partial \alpha_i} = \oint_{B_i} dS, \quad \frac{\partial \mathcal{F}}{\partial t_n} = \frac{1}{2\pi i n}\mathrm{res}_0\ \xi^{-n} dS \tag{52}$$

Their consistency follows from (47) and Riemann identities. In particular,

$$\frac{\partial^2 \mathcal{F}}{\partial t_m \partial t_n} = \frac{1}{2\pi i n}\mathrm{res}_0\ \xi^{-n}\frac{\partial dS}{\partial t_m} = \frac{1}{2\pi i n}\mathrm{res}_0\ \xi^{-n} d\Omega_m = \frac{1}{2\pi i m}\mathrm{res}_0\ \xi^{-m} d\Omega_n \tag{53}$$

From this calculation, it is clear that the definition (52) assumes that coordinates ξ are not changed under the variation of moduli. It means that they provide a moduli-independent parametrization of entire family – like w in the case of (1). Since moduli-independence of ξ should be also consistent with (32), the choice of ξ is strongly restricted: to $w^{\pm 1/N}$ in the case of (1) (see the discussion in the previous section).

The last relation in (53) (symmetricity) is just an example of the Riemann relations and it is proved by the standard argument:

$$
\begin{gathered}
0=\int d\Omega_m\wedge d\Omega_n= \\
\oint_{A_i} d\Omega_m\oint_{B_i} d\Omega_n-\oint_{A_i} d\Omega_n\oint_{B_i} d\Omega_m+\frac{1}{2\pi i}\mathrm{res}\left(d\Omega_m d^{-1}d\Omega_n\right)= \\
=0+0+\frac{1}{2\pi in}\mathrm{res}_0\ \xi^{-n}d\Omega_m-\frac{1}{2\pi im}\mathrm{res}_0\ \xi^{-m}d\Omega_n
\end{gathered}
\tag{54}
$$

where (27) and (26) are used at the final stage. The factors like n^{-1} arise since $\xi^{-n-1}d\xi=-d(\xi^{-n}/n)$. We shall also use a slightly different normalization $d\Omega_n\sim w^{n/N}\frac{dw}{w}=\frac{N}{n}dw^{n/N}$, accordingly the residues in (54) and (52) and (53) will be multiplied by N/n instead of $1/n$.

By definition, the prepotential is a homogeneous function of its arguments a_i and t_n of degree 2,

$$
2\mathcal{F}=\alpha_i\frac{\partial\mathcal{F}}{\partial\alpha_i}+t_n\frac{\partial\mathcal{F}}{\partial t_n}=\alpha_i\alpha_j\frac{\partial^2\mathcal{F}}{\partial\alpha_i\partial\alpha_j}+2\alpha_i t_n\frac{\partial^2\mathcal{F}}{\partial\alpha_i\partial t_n}+t_m t_n\frac{\partial^2\mathcal{F}}{\partial t_m\partial t_n}
\tag{55}
$$

Again, this condition can be proved with the help of Riemann identities, starting from (52), (49) and (50). At the same time, $\mathcal{F}$ is not just a quadratic function of a_i and t_n, a non-trivial dependence on these variables arise through the dependence of $d\omega_i$ and $d\Omega_n$ on moduli (like u_k or h_k) which in their turn depend on a_i and t_n. This dependence is obtained, for example, by substitution of (48) into (47):

$$
d\hat{\Omega}_n+t_m\frac{\partial d\hat{\Omega}_m}{\partial u_l}\frac{\partial u_l}{\partial t_n}=d\Omega_n,
\tag{56}
$$

i.e.

$$
\left(\sum_{m,l}t_m\frac{\partial u_l}{\partial t_n}\right)\oint_{A_i}\frac{\partial d\hat{\Omega}_m}{\partial u_l}=-\oint_{A_i}d\hat{\Omega}_n
\tag{57}
$$

The integral in the l.h.s. is expressed, according to (32), through the integrals of holomorphic 1-differentials, while the integral in the r.h.s. – through the periods of dS. If

$$
\oint_{A_i}\frac{\partial dS}{\partial u_l}=\oint_{A_i}dV_l=\Sigma_{il}
\tag{58}
$$

then

$$t_m \frac{\partial u_k}{\partial t_m} = \Sigma^{-1}_{ki} \alpha_i \tag{59}$$

The relations (59) can be thought of as the other form of the Whitham hierarchy.

$SU(2)$ **case.** Now let us discuss the solution (48) in the simplest $SU(2)$ case. First, we consider only two first non-zero times so that dS has a particular form

$$dS = t_0 d\hat{\Omega}_0 + t_1 d\hat{\Omega}_1 = t_0 dz + t_1 \lambda dz \tag{60}$$

Then, using formulas of the end of the previous section for the differentials $d\Omega_{1,2}$, one can easily obtain from (56) for $n = 0, 1$ the following two equations:

$$\begin{aligned} t_1 \frac{\partial u}{\partial t_1} &= -\frac{a}{\sigma}, \\ t_1 \frac{\partial u}{\partial t_0} &= -\frac{1}{\sigma} \end{aligned} \tag{61}$$

Any solution to the equations (61) evidently solves simultaneously (42), but not *vice versa*.

Let us use the manifest form of functions $a(u)$ and $\sigma(u)$ from (43):

$$\begin{aligned} t_1 \frac{\partial u}{\partial t_1} &= -\frac{8}{k^2} \frac{E(k)}{K(k)}, \\ t_1 \frac{\partial u}{\partial t_0} &= -\frac{2\pi}{k} \frac{1}{K(k)} \end{aligned} \tag{62}$$

Then, the first equation of (61) has the solution [14]

$$t_1 = c(t_0) \frac{k}{E(k)} \tag{63}$$

with arbitrary function $c(t_0)$. The solution to the both equations (61) takes the form

$$a(u) = \frac{const - t_0}{t_1} \tag{64}$$

This particular solution has to be compared with (45) with the constant function $\Phi(u)$ (since it has no inverse, the second form (44) does not exist for this concrete solution).

In order to get the general solution (45) to the Whitham hierarchy, one has to require for all higher odd times in (48) (i.e. with $n > 1$) to be non-zero constants. Then, these higher times parameterize the general function $\Phi(u)$. Indeed, how it can be easily obtained from (56), the solution to the Whitham equations with general dS (48) is given by formula (45) with

$$\Phi'(u) = \sum_{k=1} t_{2k+1}(-)^k \frac{(2k-1)!!}{2^k} u^k \tag{65}$$

Therefore, the differential dS (48) actually leads to the general solution to the Whitham hierarchy.

To complete our consideration of particular system (61), let us discuss what happens when one considers u as a function of *three* independent variables: t_0, t_1 and α. We are going to demonstrate that, in this case, we still get the same solution depending on *two* independent variables.

Since

$$\alpha \equiv \oint_A dS = t_1 a + t_0 \tag{66}$$

(cf. with eq.(64)) and using (40), (61) turns into

$$\begin{gathered} t_1 \frac{\partial u}{\partial t_1} = -\alpha \frac{\partial u}{\partial \alpha} + t_0 \frac{\partial u}{\partial \alpha}, \\ \frac{\partial u}{\partial t_0} = -\frac{\partial u}{\partial \alpha} \end{gathered} \tag{67}$$

with the solution

$$u = \Psi\left(\frac{\alpha - t_0}{t_1}\right) \tag{68}$$

where $\Psi(x)$ is an arbitrary function. Note that (64) now can be understood just as the condition of constant α.

Let us now note that $a(u)$ for the given data is some known function of u. Then, the inverse (not arbitrary!) function $u = \Psi(a)$ can be rewritten in the form (68) using (66) with independent variables α, t_0 and t_1. It means that one can not consider arbitrary function $\Psi(x)$ with the particular anzatz of zero higher times in (48), i.e. solutions to the equations (67) should be additionally constrained.

If now one forgets about the specific anzatz for dS with zero higher times and considers unconstrained dS (48), one should also consider the additonal

Whitham equation

$$\frac{\partial d\Omega_n}{\partial \alpha} = \frac{\partial d\omega}{\partial t_n} \tag{69}$$

that exactly gives rise to the equations (67)

$$\begin{aligned} \frac{\partial u}{\partial t_1} &= -a\frac{\partial u}{\partial \alpha}, \\ \frac{\partial u}{\partial t_0} &= -\frac{\partial u}{\partial \alpha} \end{aligned} \tag{70}$$

These equations are equaivalent to (42) with u being a function of $t_0 - \alpha$. It means that we again obtain the same solution (45) that describes u as a function of *two* independent variables t_1 and $t_0 - \alpha$. This perfectly matches the number $g + n - 1 = 2$ of independent variables.

6 Conclusion

Thus, we have constructed the prepotential for the Whitham hierarchy associated with the Seiberg-Witten theory. How it was already explained in [2], the Whitham equations themselves describe the dependence of effective coupling constants given by the period matrix of the Seiberg-Witten curve on bare coupling constants t_n. The prepotential also depends on characteristics of the vacuum parametrized by vev's u_k. Using this interpretation, one expects [7] that the constructed Whitham prepotential has to be associated with the generating function of topological correlators, the Whitham times being bare coupling constants of different local operators[5]. It issues

[5]Note also that, after the rescaling

$$h_k \to t_1^k h_k, \quad \mathcal{H}_k \to t_1^k \mathcal{H}_k \tag{71}$$

t_1 can be identified with Λ [7]. Then eqs. (74), (76) below at $n = 1$ are naturally interpreted in terms of the stress-tensor anomaly,

$$\ldots + \vartheta^4 \Theta^\mu_\mu = \beta \mathrm{tr}\Phi^2 = \ldots + \vartheta^4 \beta \mathrm{tr}\left(G_{\mu\nu}G^{\mu\nu} + iG_{\mu\nu}\tilde{G}^{\mu\nu}\right), \tag{72}$$

since for any operator $\mathcal{O}$

$$\frac{\partial}{\partial \log \Lambda}\langle \mathcal{O} \rangle \;=\; \langle \beta \mathrm{Tr}\Phi^2, \mathcal{O} \rangle \tag{73}$$

Analogous interpretation for $n \geq 2$ involves anomalies of W_{n+1}-structures.

the problem of calculating derivatives of the prepotential with their further comparison with the topological correlators. Indeed, the first two derivatives have been calculated in [7] and turned out to be

$$\frac{\partial \mathcal{F}}{\partial t^n} = \frac{\beta}{2\pi i n}\sum_m m t_m \mathcal{H}_{m+1,n+1} = \frac{\beta}{2\pi i n} t_1 \mathcal{H}_{n+1} + O(t_2, t_3, \ldots) \tag{74}$$

$$\frac{\partial^2 \mathcal{F}}{\partial \alpha^i \partial t^n} = \frac{\beta}{2\pi i n}\frac{\partial \mathcal{H}_{n+1}}{\partial a^i} \tag{75}$$

$$\frac{\partial^2 \mathcal{F}}{\partial t^m \partial t^n} = -\frac{\beta}{2\pi i}\left(\mathcal{H}_{m+1,n+1} + \frac{\beta}{mn}\frac{\partial \mathcal{H}_{m+1}}{\partial a^i}\frac{\partial \mathcal{H}_{n+1}}{\partial a^j}\partial^2_{ij}\log\theta_E(\vec{0}|\tau)\right) \tag{76}$$

etc. In these formulas parameter $\beta = 2N$, $m, n = 1, \ldots, N-1$ and $\mathcal{H}_{m,n}$ are certain homogeneous combinations of u_k, defined in terms of $P_N(\lambda)$:

$$\mathcal{H}_{m+1,n+1} = -\frac{N}{mn}\mathrm{res}_\infty\left(P^{n/N}(\lambda) dP_+^{m/N}(\lambda)\right) = \mathcal{H}_{n+1,m+1} \tag{77}$$

and

$$\mathcal{H}_{n+1} \equiv \mathcal{H}_{n+1,2} = -\frac{N}{n}\mathrm{res}_\infty P^{n/N}(\lambda) d\lambda \tag{78}$$

Note that the equations (74)-(76) at $n = 1$ already appeared in the context of topological theories [8]. These are evidently only the first steps in the direction that presumably should lead us to understanding of the supersymmetric gauge theories.

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