The Gluon Propagator in Analytic Perturbation Theory

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1.Introduction

Consistent theoretical determination of the effective coupling constant has become one of the most important tasks of QCD. The serious difficulty is the occurrence of the Landau singularities in the effective coupling constant in perturbation theory (PT). It is widely believed that in realistic quantum field theories these unphysical singularities are artifact of PT [1,2]. In QCD they appear at small space-like momentum. For this reason PT becomes useless for low-energy calculations. Supposedly, these infrared singularities signal that the perturbative vacuum differs from the true vacuum of the theory [3].

On the other hand, most of the suggestions of the confinement mechanism include assumptions about the infrared behaviour of the effective coupling constant. Thus, in the popular linear confinement picture singular infrared behaviour is assumed for the effective coupling constant $\bar{\alpha}_s(q^2) \sim q^{-2}$ [3,4]. It was confirmed that this behaviour is consistent with the Schwinger-Dyson equations (SDEs) of QCD [3,4].

Independently, in the literature has become popular the hypothesis that the effective coupling constant has a regular infrared behavior [5]. Several convincing arguments proposed to support the idea that the running coupling "freezes" at low energies [6]. The confinement mechanism, which may be consistent with this hypothesis has been proposed in Refs. [7]. A linearly rising potential is not assumed in this framework. One possible resolution of the "Landau-ghost" problem has been suggested long time ago in Refs.

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[1,2], where the "analytization procedure" elaborated in PT. Recently, this method has been successfully applied to the QCD running coupling [8-12]. Important observation has been made in Refs. [8]. It was found that the modified solution for $\bar{\alpha}_s(-q^2)$ has a finite infrared limit $\bar{\alpha}_s(0)$, which is stable with respect to higher order corrections.

The organization of the paper is as follows. In Sec.2 we present notation and discuss some technical aspects of analytic perturbation theory (APT). The structure of the β-function is studied in APT. We write the two-loop analytic effective coupling constant in terms of Lambert W function [13]. Thus, we avoid Newton's iteration method of solution. The exact and the approximate solutions are compared numerically in the infrared domain. In Sec.3 we apply the method to the gluon propagator in the Landau gauge. We have found that the Källen-Lehmann analyticity and the RG alone are not sufficient to uniquely determine the nonperturbative part of the anomalous dimension function for the gluon field. Two different definitions for the anomalous dimension function are considered. We discuss the possible resolution of the ambiguity. Sec. 4 contains concluding remarks.

2. The "analytization procedure" for the QCD running coupling

The β-function in QCD is given by

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s), \tag{1}$$

where $\alpha_s = \frac{g^2}{4\pi}$, g^2 is the renormalized coupling constant, and μ is the renormalization point. For simplicity, we have assumed that the quarks are massless. For convenience we choose the covariant Landau gauge. Eq. (1) defines the β-function order by order in PT. However, one can assume Eq. (1) beyond of PT. The assumption is that the non trivial β function exists with perturbative asymptotic expansion in powers of α_s

$$\beta(\alpha_s) \approx -\frac{\beta_0}{4\pi}(\alpha_s)^2 - \frac{\beta_1}{(4\pi)^2}(\alpha_s)^3 + \cdots \tag{2}$$

The first two coefficients of the series are independent of the chosen renormalization scheme. Their values are given by

$$\beta_0 = (11 - 2N_f/3), \quad \beta_1 = (102 - 38N_f/3), \tag{3}$$

with N_f being the number of flavors. The effective coupling constant is defined by

$$k^2 \frac{d}{dk^2} \bar{\alpha}_s(-k^2) = \beta(\bar{\alpha}_s(-k^2)), \tag{4}$$

$$\bar{\alpha}_s(\mu^2) = \alpha_s. \tag{5}$$

Convenient definition of the QCD scale parameter Λ has been given in Ref.[14]. The relation between conventionally defined parameter Λ_{cn} and Λ is given by $\Lambda_{cn} = (\eta)^{(\eta/2)}\Lambda$ with $\eta = \beta_1/\beta_0^2$. With this definition and with normalization (5) the solution of Eq. (4) is

$$\ln\left(\frac{-k^2}{\Lambda^2}\right) = \frac{4\pi}{\beta_0 \bar{\alpha}_s} - \frac{\beta_1}{\beta_0^2} \ln\left(1 + \frac{4\pi\beta_0}{\beta_1 \bar{\alpha}_s}\right) + \psi(\bar{\alpha}_s), \tag{6}$$

where

$$\psi(\alpha_s) = \int_0^{\alpha_s} (1/\beta(x) - 1/\beta^{(2)}(x))dx,$$

and $\beta^{(2)}(x)$ denotes the two-loop β-function. To express $\bar{\alpha}_s(-k^2)$ explicitly as a function of k^2, beyond the one-loop order, one must solve the transcendental equation (6). Usually, iteration procedure is used.

Let $\bar{\alpha}_{pt}^{(n)}(-k^2)$ be nth-order perturbative solution for $\bar{\alpha}_s$ determined by Eq. (6). Then the corresponding analytic solution is given by the Källen-Lehmann integral

$$\bar{\alpha}_{an}^{(n)}(-k^2) = \frac{1}{\pi} \int_0^{\infty} \frac{\rho^{(n)}(\sigma)}{(\sigma - k^2 - i0)} d\sigma, \tag{7}$$

where spectral function $\rho^{(n)}(\sigma)$ is the discontinuity of $\bar{\alpha}_{pt}^{(n)}(-k^2)$ along the positive k^2 axis. The main result of Refs.[8] is that the value $\bar{\alpha}_{(an)}(0)$ in APT is determined by one-loop calculation: $\bar{\alpha}_{an}^{(n)}(0) = \bar{\alpha}_{an}^{(1)}(0) = 4\pi/\beta_0 = 4\pi/(11 - 2N_f/3)$.

Using Cauchy's theorem one can represent the function (7) in the equivalent form

$$\bar{\alpha}_{an}^{(n)}(-k^2) = \bar{\alpha}_{pt}^{(n)}(-k^2) + \theta^{(n)}(-k^2), \tag{8}$$

the "nonperturbative" term $\theta^{(n)}$ compensates the unphysical contributions in $\bar{\alpha}_{pt}^{(n)}$. We shall assume that $\theta^{(n)}$ satisfies the representation

$$\theta^{(n)}(-k^2) = \frac{1}{\pi} \int_{k_L^2}^{0} \frac{\rho_\theta^{(n)}(\sigma)}{(\sigma - k^2 - i0)} d\sigma, \tag{9}$$

where $k_L^2 = -c_{(n)}\Lambda^2$, and $c_{(n)}$ is the positive real number calculable in PT, for example $c_{(1)} = c_{(2)} = 1$. To third order, in the MS scheme and for $N_f = 3$, we find that $c_{(3)} = 1.87$. The normalization condition (5) now reads

$$\bar{\alpha}_{an}(\mu^2) = \bar{\alpha}_{pt}^{(n)}(\mu^2) + \theta^{(n)}(\mu^2) = \alpha_s. \tag{10}$$

Let us consider the structure of the β-function in APT. From Eqs. (7) and (10) we find

$$\beta_{an}^{(n)}(\alpha_s) = -\frac{1}{\pi}\int_0^\infty \frac{\mu^2}{(\sigma+\mu^2)^2}\rho^{(n)}(\sigma)d\sigma = -\frac{1}{\pi}\int_0^\infty \frac{\bar{\rho}^{(n)}(s)}{\xi(s+1/\xi)^2}ds, \tag{11}$$

where $\bar{\rho}^{(n)}(s) = \rho^{(n)}(\Lambda^2 s)$ and ξ denotes the dimensionless variable $\xi = exp(-\phi) = \Lambda^2/\mu^2$. Note that $\xi \sim \exp(-4\pi/\beta_0\alpha_s)$ for $\alpha_s \to 0$. From Eq.(11) we see that $\beta(\alpha_s) \equiv \hat{\beta}(\xi)$ is analytical in the cut-ξ plane. Alternatively, from Eq.(10) we find

$$\beta_{an}^{(n)}(\alpha_s) = \partial\alpha_s/\partial ln\mu^2 = \beta_{pt}^{(n)}(\bar{\alpha}_{pt}^{(n)}(\mu^2)) + \bar{\theta}^{(n)}(\mu^2) =$$

$$\beta_{pt}^{(n)}(\alpha_s - \theta^{(n)}(\mu^2)) + \bar{\theta}^{(n)}(\mu^2), \tag{12}$$

where $\beta_{pt}^{(n)}(x) = -\sum_{k=0}^{n-1}\beta_k x^{k+2}/(4\pi)^{k+1}$, and $\bar{\theta}^{(n)}(\mu^2) = \mu^2\frac{\partial\theta^{(n)}(\mu^2)}{\partial\mu^2}$. We can rewrite Eq() in the form

$$\beta_{an}(\alpha_s) = \beta_{pt}(\alpha_s) + \beta_{np}(\alpha_s, \theta, \bar{\theta}), \tag{13}$$

here, for convenience we have used condensed notations suppressing the superscript (n) and arguments to the expressions above ($\theta \equiv \theta^{(n)}(\mu^2)$ ets.). The nonperturbative piece β_{np} can be written as a power series for α_s

$$\beta_{np}(\alpha_s, \theta, \bar{\theta}) = \sum_{k=0}^{n} B_k(\theta, \bar{\theta})(\alpha_s)^k, \tag{14}$$

where

$$B_0 = \bar{\theta}, \quad B_1 = -\frac{\partial\beta_{pt}(-\theta)}{\partial\theta},$$

$$B_k(\theta) = \frac{(-1)^k}{k!}\frac{\partial^k\beta_{pt}(-\theta)}{\partial\theta^k} + \frac{\beta_{k-2}}{(4\pi)^{k+1}} \quad \text{for} \quad k \geq 2, \tag{15}$$

From the representation (9) we see that $\theta(\mu^2)$ and $\bar{\theta}(\mu^2)$ are analytical functions in the cut ξ-plane. The cut is along the positive real ξ-axis $\xi \geq 1/c_{(n)}$. Then, from (15) it follows that the coefficients $B_k(\theta) \equiv \hat{B}_k(\xi)$ are analytical in the domain $\xi \leq 1/c_{(n)}$. In the weak-coupling limit they vanish exponentially $\hat{B}_k(\xi) \sim \xi \sim \exp(-\frac{4\pi}{\beta_0 \alpha_s})$. A detailed study of the one-loop β-function can be found in Ref. [9].

Now, let us consider the two-loop running coupling in APT more detailed. It is convenient to introduce quantities

$$\bar{a}_s(-k^2) = \frac{\beta_0}{4\pi}\bar{\alpha}_s(-k^2) \qquad and \qquad \bar{\beta}(a_s) = \frac{\partial a_s}{\partial \ln(\mu^2)} \tag{16}$$

To second order Eq. (6) reads

$$\ln(z) = \frac{1}{y(z)} - \eta \ln\left(1 + \frac{1}{\eta}\frac{1}{y(z)}\right), \tag{17}$$

where $z = -k^2/\Lambda^2$, $y(z) = \bar{a}_s^{(2)}(-k^2)$ and $\eta = \beta_1/{\beta_0}^2$ ($\eta = 64/81$ for $N_f = 3$). We shall accept the flavor condition $N_f < 8.05$, i.e. $\beta_1 > 0$. In this case the Landau singularity is occurred in the two-loop running coupling [5,6]. The transcendental equation (17) is solved by the iteration method. The solution given by one iteration is

$$y_{it}(z) = \frac{1}{\ln(z) + \eta \ln\left(1 + \frac{1}{\eta}\ln(z)\right)}. \tag{18}$$

The function (18) provides good approximation, for a small enough value of $\bar{a}_s$. Consider (18) as an analytic function in the whole complex z-plane. It has the Landau pole at $z = 1$, the logarithmic branch point at $z = exp(-\eta)$ and standard branch point at $z = 0$. The corresponding analytic solution is given by the Källen-Lehmann integral (see Refs.[8])

$$y_{it,an}(z) = \frac{1}{\pi}\int_0^\infty \frac{\bar{\rho}_{it}^{(2)}(x)}{x+z}dx = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^t}{e^t+z}\tilde{\rho}_{it}^{(2)}(t)dt, \tag{19}$$

where $\tilde{\rho}_{it}^{(2)}(t) \equiv \bar{\rho}_{it}^{(2)}(e^t) \equiv \rho_{it}^{(2)}(e^t\Lambda^2) = L_1/(L_1^2 + L_2^2)$ with

$$L_1 = \pi + \eta \arccos((\eta + t)/r), \quad L_2 = t + \eta \ln(r/\eta), \quad r = \sqrt{(\eta + t)^2 + \pi^2}.$$

Using the Lagrange Inversion Formula we see from (17) that z=1 is not a pole but the second order brunch point: $y(z) \sim \frac{1}{\sqrt{2\eta(z-1)}}$ for $z \to 1$. Thus, (18) violates the analytical properties of $y(z)$ near the point z=1. Therefore, it is not unreasonable to solve exactly Eq. (17) and to estimate accuracy of the solution (19). Note that the transcendental Eq. (17) is exactly solvable. Indeed, it is easy to convince that the solution is

$$y(z) = -\frac{1}{\eta}\frac{1}{1+W(\zeta)}, \tag{20}$$

where

$$\zeta = -\frac{1}{e}z^{-\frac{1}{\eta}}, \tag{21}$$

and W denotes the multivalued inverse of $\zeta = W\exp(W)$. That is the Lambert W function. A detailed review of the properties and applications of this interesting function can be found in [13]. In this article we use definition and notation of branches of W according the computer algebra system Maple Release 2 [1]. Now our task is to choose the suitable branches of W in Eq. (20). The solution must be regular in the cut complex z-plane. Consider the function (21) for $0 < \eta < 1$. It maps the cut z-plane onto the three-sheet Riemannian surface of ζ. Combining above consideration with known properties of the branches of W we find

$$W(\zeta) = \begin{cases} W(0,\zeta) & \text{if } |arg(z)| \le \eta\pi \\ W(-1,\zeta) & \text{if } \eta\pi \le arg(z) \le \pi \\ W(1,\zeta) & \text{if } -\pi \le arg(z) \le -\eta\pi \end{cases} \tag{22}$$

Taking into account Eqs. (7), (20) and (22), we write the Källen-Lehmann representation for the corresponding causal solution

$$y_{an}(z) = \frac{1}{\pi}\int_0^\infty \frac{\bar{\rho}^{(2)}(x)}{x+z}dx = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^t}{(e^t+z)}\tilde{\rho}^{(2)}(t)dt, \tag{23}$$

where $\tilde{\rho}^{(2)}(t) \equiv \bar{\rho}^{(2)}(e^t) \equiv \rho^{(2)}(e^t\Lambda^2)$ and

$$\tilde{\rho}^{(2)}(t) = -\frac{1}{\eta}Im\left(\frac{1}{1+W(1,\zeta_1(t))}\right), \quad \zeta_1(t) = \exp\left(-\frac{t}{\eta} - 1 + i(\frac{1}{\eta} - 1)\pi\right). \tag{24}$$

$-k^2/\Lambda^2$	$\bar{a}_{an}^{(2)}$	$\bar{a}_{an}^{(2).it}$	$-k^2/\Lambda^2$	$\bar{a}_{an}^{(2)}$	$\bar{a}_{an}^{(2).it}$	$-k^2/\Lambda^2$	$\bar{a}_{an}^{(2)}$	$\bar{a}_{an}^{(2).it}$
0	1.000	1.000	3	0.287	0.297	7	0.245	0.253
0.50	0.397	0.405	4	0.272	0.281	10	0.229	0.237
1	0.352	0.362	5	0.261	0.270	15	0.212	0.219
2	0.310	0.320	6	0.252	0.261	20	0.201	0.208

Table 1: One-iteration approximation vs exact solution

Numerical results for the exact two-loop function (23) as well as for the approximative one (19) are summarized in the Table 1. The relative error for the solution (19), is about 3% for the considered interval. This gives 8.3% error in the value of the QCD scale parameter $\Lambda^{(2)}$ for $N_f = 3$. Following Refs.[8] we may use the average

$$A(Q) = \frac{1}{Q} \int_0^Q \bar{\alpha}_s(\mu^2) d\mu$$

for comparison. For $Q = 2$ GeV we find that approximative solution (19) gives answer to much better than 0.5% accuracy. The reference values of $\alpha_{an}(M_\tau)$ are taken from Refs.[8]: $\alpha_{an}(M_\tau) = 0.36 \pm 0.02$ with $M_\tau = 1.78$ GeV. It should be noted that in the recent article [11] the transcendental Eq.(17) for the running coupling constant has been solved numerically on the physical cut. Nevertheless, exact expression (23) is itself valuable if we have in mind to explore global properties of the solution.

The procedure is not unique. Indeed, one can redefine the analytic charge according

$$\bar{\alpha}_{an}^{new}(-k^2) = \bar{\alpha}_{an}(-k^2) + \delta\bar{\alpha}_{an}(-k^2)$$

$$\text{with} \quad \delta\bar{\alpha}_{an}(-k^2) = \frac{1}{\pi} \int\limits_0^{K^2(\alpha_s,\mu)} \frac{\omega(\sigma)}{\sigma - k^2 - i0} d\sigma$$

where $\delta\bar{\alpha}_{an}$ is a "genuine" nonperturbative correction, which cannot be reproduced in APT, $\omega(\sigma)$ is a integrable function and $K(\alpha_s, \mu)$ is invariant under the RG. New analytic charge is asymptotically free and in the weak limit it reproduces correct perturbative series. In principle one can fix this ambiguity using appropriate truncated set of the SDEs. Indeed, in the recent publications [15] reasons have given to believe that $\delta\bar{\alpha}_{an}$ cannot be zero,

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[1] In Ref.[13] different conventions are accepted.

and an infrared regular QCD running coupling constant cannot be consistent with the SDEs for the gluon propagator. Ghost-free axial gauge has been chosen in this work. On the contrary, in Ref.[16] the infrared finite effective coupling constant has been realized within the framework of the SDEs in the Landau gauge. However, it was shown that the solution for the gluon propagator does not satisfy the Källen-Lehmann analyticity. We see that situation is not so clear. Nevertheless, APT can be considered as a minimal causal extension of standard PT.

3. The gluon propagator in APT

The method discussed above originally has been presented for calculating modified propagators within framework of PT [1]. However, the author of [1] did not explore renormalization invariance of the theory. This step was taken in paper [2], where the procedure has been formulated for the effective coupling constants in renormalizable theories. The photon propagator was considered using this method. In QED due to the Ward identity simple relation holds

$$\gamma_v(\alpha) = \beta(\alpha)/\alpha, \tag{25}$$

here $\gamma_v(\alpha)$ is the anomalous dimension of the gauge field. Because of (25) the QED effective coupling is proportional to the transverse dimensionless structure of the photon propagator. In QCD the relation (25) generally is not valid [2]. Therefore, the method can not be applied to the gluon propagator unambiguously . Let us consider the gluon propagator in the Landau gauge $D_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu/k^2)D(-k^2)$. We assume that quarks are massless and normalize the propagator at the Euclidean point $k^2 = -\mu^2$

$$D(-k^2) = -d(-k^2/\mu^2, \alpha_s)/k^2, \qquad d(1, g^2) = 1. \tag{26}$$

Invariance under the RG, in the Landau gauge, leads

$$d(-k^2/\mu^2, \alpha_s) = \exp\left(\int_{\alpha_s}^{\bar{\alpha}_s(-k^2)} \frac{\gamma_v(x)}{\beta(x)} dx\right) \tag{27}$$

The formal power series for the anomalous dimension function γ_v, in the Landau gauge, is given by

$$\gamma_v(\alpha_s) = -(\frac{\gamma_0}{4\pi}\alpha_s + \frac{\gamma_1}{16\pi^2}{\alpha_s}^2 + \ldots), \tag{28}$$

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[2] (25) is still valid in axial-type gauges, as well as, in the background field gauge.

with $\gamma_0 = (13 - 4N_f/3)/2$ and $\gamma_1 = (531/8 - 61N_f/6)$.

A detailed investigation of the gluon propagator in the framework of analyticity and asymptotic freedom has been undertaken in Refs.[17]. The gluon propagator satisfies an unsubtracted spectral representation

$$D(k^2, \alpha_s, \mu) = \frac{1}{\pi} \int\limits_0^\infty \frac{\rho_v(\sigma, \alpha_s, \mu)}{\sigma - k^2 - i0} d\sigma. \tag{29}$$

For a limited number of flavors $N_f \leq 9$ in the Landau gauge the weight function ρ_v satisfies the superconvergence relation $\int\limits_0^\infty \rho_v(\sigma, \alpha_s, \mu) d\sigma = 0$. It was argued in Refs.[17] that For $N_f \leq 9$ there is an renormalization invariant point $K^2(\alpha_s, \mu)$ such that ρ_v is a negative measure for $\sigma \geq K^2$. We can start from formula (27). The solution for the β-function is known in APT [8]. But, the anomalous dimension function γ_v is not determined within this method yet. One possible way is to assume following relation

$$\gamma_{v,an}^{(n)}(\alpha_s)/\beta_{an}^{(n)}(\alpha_s) = \gamma_v^{(n)}(\alpha_s)/\beta^{(n)}(\alpha_s), \tag{30}$$

where $\beta^{(n)}$ and $\gamma_v^{(n)}$ are standard perturbative expressions for the renormalization group functions in the n-loop order, and $\beta_{an}^{(n)}$ and $\gamma_{v,an}^{(n)}$ are corresponding quantities in APT. Then from formula (27), in the one-loop order, we may write

$$D_{an}^{(1)}(-k^2) = -\frac{1}{k^2} \left(\frac{\bar{\alpha}_{an}^{(1)}(-k^2)}{\alpha_s} \right)^{\gamma_0/\beta_0} = -\frac{1}{k^2} c_v \left(\frac{1}{\ln(-k^2/\Lambda^2)} + \frac{1}{1 + k^2/\Lambda^2} \right)^{\gamma_0/\beta_0}, \tag{31}$$

with $c_v = (\frac{4\pi}{\beta_0 \alpha_s})^{\gamma_0/\beta_0}$. It is easy to convince that expression (31) has correct analytical properties, since, $\bar{\alpha}_{an}^{(1)}(-k^2)$ is analytic and it does not vanish in the finite part of the complex k^2-plane. Thus, function (31) satisfies the Källen-Lehmann representation. The corresponding weight function is given by

$$\rho_v^{(1)}(\sigma) = \rho_+(\sigma) + \rho_-(\sigma), \tag{32}$$

where ρ_+ (ρ_-) is positive (negative) measure, $\rho_+(\sigma) = c_v \pi \delta(\sigma)$ and

$$\rho_-(\sigma) \equiv \bar{\rho}_-(\bar{\sigma}) =$$

$$-c_v \frac{1}{\bar{\sigma}\Lambda^2} [R(\bar{\sigma})]^{\frac{\gamma_0}{\beta_0}} \sin \left[\frac{\gamma_0}{\beta_0} \arccos \left(\frac{1}{R(\bar{\sigma})} \left(\frac{\ln \bar{\sigma}}{\ln^2 \bar{\sigma} + \pi^2} + \frac{1}{\bar{\sigma} + 1} \right) \right) \right],$$

here $\bar{\sigma} = \sigma/\Lambda^2$ and

$$R(\bar{\sigma}) = \sqrt{\left(\frac{\ln\bar{\sigma}}{\ln^2\bar{\sigma}+\pi^2} + \frac{1}{1+\bar{\sigma}}\right)^2 + \left(\frac{\pi}{\ln^2\bar{\sigma}+\pi^2}\right)^2}.$$

The superconvergence relation in this case reads $\int\limits_0^\infty \rho_-(\sigma)d\sigma = -\pi c_v$. For the RG invariant scale $K^2(g,\mu^2)$, which has been introduced in Refs.[17], the solution (31) predicts trivial value $K^2(g,\mu^2) = 0$. The weight function (32) has integrable singularity at $\sigma = 0$.

For the two-loop gluon propagator similar procedure leads

$$d^{(2)}_{an}(-k^2) = \bar{d}^{(2)}(\bar{\alpha}^{(2)}_{an}(-k^2))/\bar{d}^{(2)}(\alpha_s), \qquad (33)$$

where $\bar{d}^{(2)}(\alpha) = (\alpha)^{\gamma_0/\beta_0}(1+\alpha\beta_1/4\pi\beta_0)^{-(\gamma_0/\beta_0-\gamma_1/\beta_1)}$, it is evident that (33) satisfies requirement of analyticity, since, the equation $\bar{\alpha}^{(2)}_{an}(-k^2) = -4\pi\beta_0/\beta_1 < 0$ has no solutions in the cut complex k^2-plane. The reason is that $\bar{\alpha}^{(2)}_{an}$ itself satisfies the Källen-Lehmann representation with positive defined weight function. One can assume the relations similar of (30) for the quark and the ghost fields too, and by the way, write analytic solutions for the corresponding propagators.

Now we suggest different method for restoring analyticity and renormalization invariance in perturbation theory. In this case we do not accept the relation (30). Instead, we start from the perturbative expression for the gluon propagator. To the one-loop approximation, in the Landau gauge, we have

$$d^{(1)}(-k^2/\mu^2,\alpha_s) = \frac{1}{1+\frac{\alpha_s}{4\pi}\gamma_0\ln(-\frac{k^2}{\mu^2})} = \left(\frac{\beta_0}{\gamma_0 a_s}\right)\frac{1}{\ln(\frac{-k^2}{\mu^2}) - \frac{\beta_0}{\gamma_0}\ln(\xi)}, \qquad (34)$$

here γ_0 is given by (28), $a_s = \frac{\beta_0}{4\pi}\alpha_s$, $\xi = \Lambda^2/\mu^2$ and we have used standard perturbative definition of the Λ parameter $\Lambda^2 = \mu^2 e^{-\frac{1}{a_s}}$. The "analitization procedure" for (34) leads

$$d^{(1)}_{an}(u,\alpha_s) = \bar{d}^{(1)}_{an}(u,\alpha_s)/\bar{d}^{(1)}_{an}(1,\alpha_s) \qquad (35)$$

where

$$\bar{d}^{(1)}_{an}(u,\alpha_s) = \left(\frac{1}{\ln u - \frac{\beta_0}{\gamma_0}\ln\xi} - \frac{\xi^{\frac{\beta_0}{\gamma_0}}}{u - \xi^{\frac{\beta_0}{\gamma_0}}}\right), \qquad (36)$$

and $u = -k^2/\mu^2$. In order to preserve the normalization (see (26)) we have introduced in (35) the necessary nonperturbative factor. Furthermore, the invariant scale Λ in (35) now is determined according APT, since, the procedure implies the replacement $\Lambda_{pt} \to \Lambda_{an}$. Thus, to first order according APT [8], we have the relation

$$a_s = \frac{\beta_0}{4\pi}\alpha_s = -\frac{1}{\ln\xi} - \frac{\xi}{1-\xi} = \frac{1}{\phi} + \frac{1}{1-e^{\phi}}$$

where $\phi = -\ln\xi$. Using (35) we find the anomalous dimension for the gluon field

$$\gamma_v^{(1)}(\alpha_s) \equiv \bar{\gamma}_v^{(1)}(\phi) = \lim_{u\to 1} u\frac{\partial}{\partial u}\ln d_{an}^{(1)}(u,\alpha_s) = \frac{-\frac{(\gamma_0/\beta_0)^2}{\phi^2} + \frac{e^{-(\beta_0/\gamma_0)\phi}}{(1-e^{-(\beta_0/\gamma_0)\phi})^2}}{\frac{(\gamma_0/\beta_0)}{\phi} - \frac{e^{-(\beta_0/\gamma_0)\phi}}{1-e^{-(\beta_0/\gamma_0)\phi}}}. \quad (37)$$

The expression (35) does not obey renormalization invariance. We can consider (35) as a known estimate and then use Eq (27) to derive the improved solution. Taking into account the identity

$$\int_{\alpha_s}^{\bar{\alpha}_s(-k^2)} \frac{\gamma_v(x)}{\beta(x)}dx = \int_{\ln(\mu^2/\Lambda^2)}^{\ln(-k^2/\Lambda^2)} \bar{\gamma}_v(\phi)d\phi,$$

from Eqs. (27) and (37) we obtain

$$d_{im}^{(1)}(-k^2/\mu^2,\alpha_s) = exp\left(-\frac{\gamma_0}{\beta_0}\int_{\frac{\beta_0}{\gamma_0}\ln(\mu^2/\Lambda^2)}^{\frac{\beta_0}{\gamma_0}\ln(-k^2/\Lambda^2)} \frac{(e^t-1)^2 - t^2e^t}{t(e^t-1)(e^t-1-t)}dt\right). \quad (38)$$

The integral in (38) can be performed explicitly. This yields

$$d_{im}^{(1)}(-k^2/\mu^2,\alpha_s) = \bar{d}_{im}^{(1)}(-k^2/\Lambda^2)/\bar{d}_{im}^{(1)}(\mu^2/\Lambda^2), \quad (39)$$

where

$$\bar{d}_{im}^{(1)}(-k^2/\Lambda^2) = \left(\frac{1}{\frac{\beta_0}{\gamma_0}\ln(-k^2/\Lambda^2)} - \frac{1}{(-k^2/\Lambda^2)^{\frac{\beta_0}{\gamma_0}} - 1}\right)^{(\gamma_0/\beta_0)}. \quad (40)$$

We see that function (39) satisfies the Källen-Lehmann analyticity for $\gamma_0/\beta_0 \geq 0.5$. The corresponding flavor condition is $N_f \leq 3$. For $N_f > 3$ the unphysical singularities appear in the first Riemann sheet. This limitation for N_f turns out to be natural. Since, the singularities in (40) occurred at $|k^2/\Lambda^2| = 1$ where the number of active quarks are just three. For this reason we cannot reject the solution (40) using arguments of analyticity [3].

Thus, we need independent criterion to select the suitable analytic solution. Recently Such a criterion has been suggested in Ref.[15], where the principle of minimality for the nonperturbative contributions in perturbative (ultraviolet) region has been formulated. Using this principle one can easily verify that the solution (39) is preferable. Indeed, (39) predicts more rapid decrease of the nonperturbative contributions in ultraviolet region then the solution (31) provided that $66/39 \leq \beta_0/\gamma_0 \leq 2$ for $0 \leq N_f \leq 3$.

4. Conclusion

The procedure does not destroy renormalizability of the theory. The only effect is that the renormalization constants receive finite nonperturbative corrections. This was demonstrated by explicit calculations of the β-function and the anomalous dimension function of the gluon field.

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[3] The author owes to D.V. Shirkov for drawing his attention to this peculiarity of the solution (40).

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