Asymptotics of Pion Form Factor in Relativistic Hamiltonian Dynamics and Quark Counting Rules.

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Abstract

The asymptotics of pion charge form factor is obtained in the framework of relativistic Hamiltonian dynamics for the infinite value limit of momentum transfer and zero value limit of constituent–quark mass. It is shown that this asymptotics is the same as given by the quark counting rules and is determined by relativistic kinematics only, being independent on the constituent quarks interaction in pion.

It is now well established that the description of the electromagnetic structure of composite quark systems at low and intermediate momentum transfers, i.e. in the region of so called "soft" processes, needs nonperturbative approaches. Moreover, in the investigation of composite systems of light quarks (e.g., the pion) one has to take into account relativistic effects, which give essential contribution at low momentum transfers.

During last years the relativistic Hamiltonian dynamics (RHD) is widely used in this nonperturbative region for the investigation of composite quark systems (the foundation of the approach, references to calculations and some historical aspects can be found in the reviews [1]– [3], see also [4] – [9]). The RHD can be formulated in different ways. The most widely used forms are: the instant form dynamics, the point form dynamics and the light–front dynamics. At present time the most popular is light–front dynamics in the framework of which the many calculations of the hadronic properties were performed. In a number of our works the instant form is used successfully for the description of electroweak properties of the composite systems [4] – [9]. It is important to remind ourselves that the RHD is not QCD but very successful phenomenological model. The main feature of RHD *versus* field theories is the extraction of the finite number of degrees of freedom in each concrete case. The establishment of the relation between RHD and fundamental principals of description of the quark composite systems is a important and as yet unresolved problem. This is especially true in regard to the principals of Quark Counting Rules which establish the asymptotics at large transferred momenta for hadron form factors in fact by the model independent way (see [10] for the details):

$$F_a(Q^2) \quad \sim \quad \left(\frac{1}{Q^2}\right)^{(n_a-1)} , \quad Q^2 \to \infty \tag{1}$$

where $Q^2 = -\,t = -\,q^2$, q – transferred momentum, n_a – is a number of the constituent quarks in basic configuration of hadron.

For the pion the asymptotics has the form:

$$F_\pi(Q^2) \quad \sim \quad Q^{-2}\,. \tag{2}$$

Quark Counting Rules are based on the rather general physical assumptions:
– composite quark structure of hadrons (hypothesis of quark confinement);
– quasi–free behaviour of quarks bound inside hadrons (asymptotic freedom of QCD);
– revelation of point–like structure of hadrons in exclusive hard collisions (approximate scale invariance);
– decoupling of the effects of the small and large distances in hard processes (factorization hypothesis).
These assumptions lead to formulation of the general principle in the high energy hadron physics so called "the principle of the automodel (or self–similar) behaviour".

In the present paper we show that the asymptotics (2) can be obtained in the frame of RHD instant form [4] if $Q^2 \to \infty$, $M \to 0$ (where M is the constituent quark mass), and that the asymptotic behavior has universal character: it does not depend on the choose of concrete interaction model for constituent quarks and is defined by the relativistic kinematics of two–quark system only.

The limits defined above are connected with the following meaning of the physical behavior at large momentum transfers. At $Q^2 \to \infty$ the quarks become free and their mass becomes equal to the initial one (current mass). For the pion with light quarks this means in fact that $M \to 0$. If a dimension parameter b defines the characteristic hadron scale (the confinement scale), then the limits have the following meaning: $Q^2 \gg b,\ M \ll b$.

In our approach the electromagnetic pion form factor is represented in the form of a functional, generated by the so called free two–particle form factor [4]. This free form factor describes the electromagnetic properties of the system of two free particles, and in the case of pion it defines the electromagnetic current matrix element for the system of two free quarks. If the free quarks have the quantum numbers corresponding to pion, then the matrix element has the form [4]:

$$< \vec{P}, \sqrt{s}, J, l, S, m_J |\, j_\mu \,|\vec{P}', \sqrt{s'}, J', l', S', m_J{}' >=$$

$$= A_\mu(s, Q^2, s')\, g_0(s, Q^2, s')\,. \tag{3}$$

Here $Q^2 = -(P-P')^2$, $s = P^2$ – system invariant mass square, $J = S = l = 0$, $g_0(s, Q^2, s')$ – the free two–particle form factor for the system of particles without interaction. The vector $A_\mu(s, Q^2, s')$ is determined by the current transformation properties (Lorentz–covariance and current conservation law) [4]. The explicit form of the function $g_0(s, Q^2, s')$ in the case of point quarks was obtained in [11]:

$$g_0(s, Q^2, s') = \frac{(s + s' + Q^2)Q^2}{2\sqrt{(s-4M^2)(s'-4M^2)}}\,\frac{\theta(s, Q^2, s')}{[\lambda(s, -Q^2, s')]^{3/2}}\,\frac{1}{\sqrt{1 + Q^2/4M^2}}\times$$

$$\times \left\{(s + s' + Q^2)\cos(\omega_1 + \omega_2) + \frac{1}{M}\xi(s, Q^2, s')\,\sin(\omega_1 + \omega_2)\right\}\,. \tag{4}$$

Here $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc), \xi = \sqrt{ss'Q^2 - M^2\lambda(s, -Q^2, s')}$, ω_1 and ω_2 - the parameters of Wigner rotation:

$$\omega_1 = \mathrm{arctg}\frac{\xi(s, Q^2, s')}{M[(\sqrt{s} + \sqrt{s'})^2 + Q^2] + \sqrt{ss'}(\sqrt{s} + \sqrt{s'})},$$

$$\omega_2 = \mathrm{arctg}\frac{(2M+\sqrt{s}+\sqrt{s'})\xi(s,Q^2,s')}{M(s+s'+Q^2)(2M+\sqrt{s}+\sqrt{s'})+\sqrt{ss'}(4M^2+Q^2)}\ . \qquad (5)$$

$\theta(s,Q^2,s') = \vartheta(s'-s_1) - \vartheta(s'-s_2)$, ϑ - the step function,

$$s_{1,2} = 2M^2 + \frac{1}{2M^2}(2M^2+Q^2)(s-2M^2) \mp \frac{1}{2M^2}\sqrt{Q^2(Q^2+4M^2)s(s-4M^2)}.$$

In fact, the form factor $g_0(s,Q^2,s')$ in (3) is a regular distribution (generalized function) and is a linear continuous functional on a test functions space [12].

To be used for pion form factor the functional given by $g_0(s,Q^2,s')$ is to be defined on the space of functions $\varphi(s,s') = \psi(s)\,\psi(s')$ using the following prescription [4]:

$$F_\pi(Q^2) = \int d\sqrt{s}\; d\sqrt{s'}\;\psi(s)\, g_0(s,Q^2,s')\,\psi(s') \qquad (6)$$

$\psi(s)$, $\psi(s')$ are functions normalized with the relativistic density of states: $\psi(s) = \sqrt[4]{s}\,u(k)\,k$; $u(k)$ are the two–quark wave functions of the relative motion S – state, $s = 4(k^2+M^2)$.

So, to obtain the asymptotic behavior of the pion form factor one has to evaluate the asymptotics at $Q^2 \to \infty$, $M \to 0$ of the generalized function $g_0(s,Q^2,s')$. This function defined on a certain class of functions gives the pion form factor.

The constituent–quark mass M and the momentum–transfer square Q^2 are the parameters of generalized function $g_0(s,Q^2,s')$. So the limits $Q^2 \to \infty$ and $M \to 0$ are to be considered in the generalized sense (in the sense of distributions) as the limits of generalized functional. The possibility of realizing the calculation of this limit for the generalized function $g_0(s,Q^2,s')$ can be justified physically by the fact that the pion in the asymptotic region can be considered as a free quark system with pion quantum numbers and the electromagnetic properties of such system are defined just by the free two–particle form factor $g_0(s,Q^2,s')$.

To obtain the asymptotic expansion of the generalized function we shall make the following assumptions.

1. Let the generalized function be given on the space R^2. During the calculation of (6) we have not used such assumption, in (6) the variables s , s' were in the physical region s , $s' \geq 4M^2$.

2.For the test function space let us take as usually the space $\mathcal{S}(R^2)$ of infinitely differentiable functions, decreasing at $|\vec{s}| \to \infty$ ($\vec{s} = (s, s')$) with all its derivatives faster than any degree of $1/|\vec{s}|$ [12]. This means that test functions decrease not slowlier than $\exp(-\,\beta\,|\vec{s}|^{\alpha})\,, \alpha > 0$.

3.Let us define the functional giving the regular generalized function by

$$< \tilde{g}_0(s, Q^2, s')\,, \varphi(s, s') >= \int d\mu(s, s')\,\tilde{g}_0(s, Q^2, s')\,\varphi(s, s')\,. \tag{7}$$

Here

$$\tilde{g}_0(s, Q^2, s') = 16\,\vartheta(s - 4M^2)\,\vartheta(s' - 4M^2)\,g_0(s, Q^2, s') \tag{8}$$

$$d\mu(s, s') = \sqrt[4]{ss'}\,d\mu(s)\,d\mu(s')\,, \quad d\mu(s) = \frac{1}{4}k\,d\sqrt{s}\,.$$

$\vartheta(x)$ – the standard step function. If $\varphi(s, s') = u(k(s))\,u(k'(s'))$ then the functional (7) has the form (6).

In fact the calculation of the asymptotics of the generalized function is just the calculation of the asymptotics of the functional (7) when $Q^2 \to \infty\,, \quad M \to 0$.

To estimate the asymptotics of the integral in (7) let us realize the following steps:

1. Let us transform the integral in s' to the integral on the segment [0,1]. To perform this transformation we use the fact that the free two–particle form factor $g_0(s, Q^2, s')$ contains the cutting functions

$$\vartheta(s' - s_1(s, Q^2)) - \vartheta(s' - s_2(s, Q^2))\,.$$

2. Let us introduce two dimensionless parameters and let us realize the limiting procedure keeping $\eta = Q^2/b^2 \to \infty$, $\xi = b/2M \to \infty$.

3. In the integral in s let us introduce the new variable $y = E/2M$, $s = 4M^2 + 2ME$, and let us separate explicitly the dependence on the parameter η introduced above.

4. Let us restrict ourselves by the main term of the integrand in the functional (7) when $\eta \to \infty$. This can be done because the integrals do converge uniformly.

With help of the theorems from [13] we obtain the asymptotic expansion of (7) at $\eta\,,\ \xi \to \infty$. Up to a preasymptotic factor this result has following form (see [9] for the details):

$$< \tilde{g}_0(s, Q^2, s')\,, \varphi(s, s') > \quad \sim \quad < \frac{1}{Q^2}\,g(s, s')\,, \varphi(s, s') > \,. \tag{9}$$

In the Eq.(9) the function $g(s,s')$ is given by:

$$g(s,s') = M^4\,\frac{1}{\sqrt[4]{s(s-4M^2)}}\frac{1}{(s-2M^2)^{5/4}}\frac{\sqrt{s}+\sqrt{s-4M^2}}{(\sqrt{s}-\sqrt{s-4M^2})^2}\cdot$$

$$\cdot\vartheta(s-4M^2)\,\delta(\mu(s')-\mu(s))\,.$$

Finally, we proved that in the sense of distributions the following asymptotic estimate is valid:

$$\tilde{g}_0(s,Q^2,s')\;\sim\;\frac{1}{Q^2}\,g(s,s')\,,\quad Q^2\;\to\infty\,,\quad M\;\to\;0\,. \qquad (10)$$

Let us note that this result is also valid in the case when the test function space is larger than $\mathcal{S}(R^2)$. It is necessary only that the test functions ensure the uniform convergence of the integrals entering the functional (7).

Applying the result (10) to the Eq.(6), one can see that the asymptotics does not depend on the choose of the functions $u(k)$, that is on the concrete form of the quark interaction in pion.

Thus, we obtained the pion form factor asymptotics which 1) coincides with Quark Counting Rules asymptotics (2), 2) does not depend on the interaction model for quarks in pion, 3) is defined by the free two–particle form factor considered as generalized function. This free two–particle form factor is obtained through relativistic kinematics method, so the asymptotics is due to the relativistic kinematics.

Let us compare our result with the results obtained by other approaches. In the frame of light front RHD [14] the asymptotic behavior of pion form factor (2) when $Q^2\to\infty$, $M\to 0$ was obtained in the case of special power form restriction for the wave function asymptotics. In papers [15], [16] the asymptotics of the form (2) was found for the gaussian wave function.

There is a principal difference between our results and the results of above mentioned papers. In those papers the results are the cosequence of the structure of the argument of the wave function for interacting quarks and of the special limitations for wave function asymptotics. It seems difficult to interpret this fact from the asymptotic freedom point of view which in fact is the base of well known result (2). So the behavior of the two–quark system form factor in the range $Q^2\to\infty$, where quarks are free, appears to be a kind of "memory" about the quark interaction.

In our approach the result coinciding with (2) follows directly from relativistic kinematics which is valid for all energies. Let us emphasize that if we neglect, for example, the relativistic spin rotation effect (if we let the rotation parameters $\omega_{1,2}$ in $g_0(s, Q^2, s')$ be zero) then the asymptotics of the generalized function in (10) is immediately changed. This is in contrast with the case of light front dynamics. It is easy to show that the analog of the relativistic spin rotation in the light front dynamics – Melosh rotation [15] – does not affect the pion form factor asymptotics.

Let us give now the results for pion form factor asymptotics in the cases of harmonic oscillator model [15]:

$$u(k) = N_{HO} \exp(-k^2/2b^2)\,. \tag{11}$$

and for power law wave function [17]:

$$u(k) = N_{PL}\,(k^2/b^2 + 1)^{-n}\,. \tag{12}$$

The gaussian form of wave function makes the problem of the calculation of asymptotics much simpler because in this case one can use the theorems from [13].

We use the model (11) to demonstrate the role of relativism and, in particular, the role of the relativistic spin rotation effect. The estimate for the pion form factor including relativistic spin rotation effect gives

$$F_\pi(Q^2) \;\sim\; 32\sqrt{2}\frac{b^2}{Q^2}\,, \tag{13}$$

while without relativistic spin rotation:

$$F_\pi(Q^2) \;\sim\; 4\sqrt{2}\frac{M}{Q}\,. \tag{14}$$

The contribution of the relativistic spin rotation is essential. If we neglect this contribution, we do not obtain correct (i.e.coinciding with Quark Counting Rules) asymptotics.

To clear up the role of relativism in our calculation let us write the nonrelativistic asymptotics obtained from the asymptotics estimate of nonrelativistic limit of Eq. (6) in the case of the model (11):

$$F_\pi(Q^2) \sim \;\exp\left(-\,\frac{Q^2}{16\,b^2}\right)\,. \tag{15}$$

The nonrelativistic asymptotics gives Gaussian decrease for the form factor. The comparison of (13) and (14) with (15) emphasizes the role of relativism. Let us note, that in (13)– (15) the explicit form of the normalization constant in (11) is taken into account: $N_{HO} = (4\,/\sqrt{\pi}\,b^3)^{1/2}$.

In the case of the model (12) the form–factor asymptotics has the form:

$$F_\pi(Q^2)\;\sim\;2\frac{b^2}{Q^2}\left(b^3\,N_{PL}^2\right)\frac{\Gamma\left(\frac{5}{4}\right)}{n^{5/4}}\,B\left(\frac{5}{4}\,,\,n-\frac{5}{4}\right)\;,\quad n\;>\;\frac{5}{4} \tag{16}$$

Here $B(x\,,\,y), \Gamma(x)$ are Euler beta– and gamma–functions respectively. Note, that (16) shows that the asymptotics (in the sense of distributions) form (10) is really valid for the test function space larger than $\mathcal{S}(R^2)$.

Let us note, that the asymptotics forms (13) and (16) do not contain the dimension parameter of quark mass. This fact is connected with the physics of asymptotically large momentum transfers when all scales are determined by the confinement scale. In the models (11) and (12) it is b that plays in fact the role of such parameter.

To conclude, let us formulate our main results. 1) In the frame of instant form relativistic Hamiltonian dynamics the asymptotics at $Q^2 \to \infty$, $M \to 0$ of the pion form factor coincides with that given by Quark Counting Rules. 2) The asymptotics is determined only by the relativistic kinematics, in particular, by the relativistic spin rotation effect, and does not depend on the choose of quark wave function in pion, i.e. on the model for quark interaction.

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