Liouville Field Theory and 2d Gravity

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Abstract

We investigate character of physical degrees of freedom for the models of 2d gravity and study it's dependence on the topology of space-time manifold.

I Introduction

The Einstein-Hilbert action in 2-dimensions does not lead to dynamical equations for the metric tensor $g_{ab}(X)$ $(X := (x^0, x^1)$; (a and $b = (0, 1)$) [1]. This degeneracy is related to the fact that the corresponding Lagrangian $\mathcal{L}_0 = \sqrt{-g}\ R(g_{ab})$ is a 'total derivative' and the action is expressed only by the 'surface terms'.

For the conformal gauge [2]

$$g_{ab}(X) = \exp q(X) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1}$$

the scalar curvature $R(X)$ takes the form

$$R(X) = e^{-q(X)}(\partial_1^2 - \partial_0^2)q(X). \tag{1.2}$$

and the degeneracy of the Einstein-Hilbert Lagrangian becomes apparent.

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One can specify a model of 2d gravity by requiring that the field $q(X)$ satisfies some dynamical equation. The Liouville field equation [3]

$$(\partial_0^2 - \partial_1^2)\, q(X) + \Lambda e^{q(X)} = 0 \tag{1.3}$$

(where Λ is a non-zero constant) is usually considered as a model of 2-dimensional general relativity [2], [4]. The field $q(X)$ satisfying (1.3) leads to the manifold with a constant curvature $R(X) = \Lambda$ (see (1.2)). The Liouville equation arises in a large variety of problems of mathematical physics due to the conformal invariance of this equation. General solution of (1.3) has the form [3]

$$q(x^+, x^-) = \log \frac{8A^{+\,\prime}(x^+)A^{-\,\prime}(x^-)}{|\Lambda|\,[A^+(x^+) + \epsilon A^-(x^-)]^2} \tag{1.4}$$

where $x^\pm = x^1 \pm x^0$ are the light cone coordinates, ϵ is a sign of Λ ($\epsilon = \Lambda/|\Lambda|$) and $A^\pm$ are any functions with $A^{\pm\,\prime} > 0$.

The standard action for the scalar field in Minkowski space, which leads to the Liouville equation (1.3) has a non-covariant form from the point of view of general relativity. To obtain the Liouville equation (as the equation for the metric tensor in the conformal gauge) from the covariant action, it is necessary to introduce some additional auxiliary fields. For a model with a reparametrization invariant action and auxiliary fields, usually it is difficult to guess what kind of physical degrees of freedom the system has and what are its physical variables [5, 6]. Construction of gauge invariant physical variables is the most important problem of 4-dimensional Einstein general relativity as well. In this note we study this problem for the simple models of 2d gravity, using the conformal invariance of the Liouville theory.

II 2d gravity with the dilaton field

We start with the action [4]

$$S = \int d^2x \sqrt{-\hat{g}}\ \left[\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \phi R(\hat{g}_{ab}) - \Lambda e^{\phi}\right], \tag{2.1}$$

where Λ is a 'cosmological constant', $\hat{g} := det\ \hat{g}_{ab}$, $\hat{g}^{ab}$ is the inverse matrix to the fiducial metric tensor $\hat{g}_{ab}$, $R(\hat{g}_{ab})$ denotes the corresponding scalar curvature and ϕ is a scalar filed, which is known as the Liouville mode, or the dilaton field.

The physical metric tensor g_{ab} is related to $\hat{g}_{ab}$ by

$$g_{ab} = e^{\phi}\hat{g}_{ab} \tag{2.2}$$

and the action (2.1) takes the form

$$S = \int d^2x\sqrt{-g}\ \left[-\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \phi R(g_{ab}) - \Lambda\right], \tag{2.3}$$

where we have used the relation of scalar curvatures in 2-dimensions for conformally related metrics (2.2)

$$R(\hat{g}_{ab}) = e^{\phi}\ [\nabla^2\phi + R(g_{ab})] \tag{2.4}$$

and we have neglected the surface term $\partial_a(\phi\sqrt{-g}\ g^{ab}\partial_b\phi)$, using the form of the Beltrami operator ∇^2 for the metric g_{ab}.

Action (2.3) defines the following 'dynamical' equations

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi} = \nabla^2\phi + R(g_{ab}) = 0, \tag{2.5}$$

$$-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = \partial_a\phi\partial_b\phi + g_{ab}\left(-\frac{1}{2}g^{cd}\partial_c\phi\partial_d\phi - \Lambda\right) - 2(g_{ab}\nabla^2 - \nabla_a\nabla_b)\phi = 0. \tag{2.6}$$

Taking trace of (2.6) and using (2.5) we get that the dynamical system (2.3) describes a space-time manifold with a constant curvature $R(g_{ab}) = \Lambda$.

For the light cone coordinates $x^{\pm} = x^1 \pm x^0$ the conformal metric (1.1) takes the form:

$$g_{+,+} = g_{-,-} = 0, \qquad g_{+,-} = g_{-,+} = -\frac{1}{2}e^q$$

and from (2.5)-(2.6) we obtain

$$\partial^2_{+,-}\varphi = 0, \qquad 4\partial^2_{+,-}q = \Lambda e^q, \tag{2.7}$$

$$(\partial_+\varphi)^2 - 2\partial^2_{+,+}\varphi = (\partial_+q)^2 - 2\partial^2_{+,+}q, \qquad (\partial_-\varphi)^2 - 2\partial^2_{-,-}\varphi = (\partial_-q)^2 - 2\partial^2_{-,-}q, \tag{2.8}$$

where $\varphi = q - \phi$.

As it was expected the field $q(X)$ satisfies the Liouville equation (1.3), while $\varphi(X)$ is a free field. According (2.8), the traceless energy-momentum tensors of these two fields are equal to each other [2, 7, 8].

One can integrate equations (2.8) and express the Liouville field $q(X)$ as a functional of the free field $\varphi(X)$. After integration we get

$$q(x^+, x^-) = \log \frac{8F'_+(x^+)F'_-(x^-)}{|\Lambda|\left[\alpha F_+(x^+)F_-(x^-) + \beta_+F_+(x^+) + \beta_-F_-(x^-) + \gamma\right]^2}, \quad (2.9)$$

where

$$F_\pm(x^\pm) = \int_{x_0^\pm}^{x^\pm} d\xi^\pm \exp\left(\int_{x_0^\pm}^{x^\pm} d\eta^\pm \partial_\pm \varphi(\eta^\pm)\right), \quad (2.10)$$

(x_0^+, x_0^-) are the coordinates of a fixed point on the space-time, and the integration constants $(\alpha, \beta_\pm, \gamma)$ satisfy the condition

$$\beta_+\beta_- - \alpha\gamma = \epsilon \qquad (\epsilon := \Lambda/|\Lambda|). \quad (2.11)$$

The latter defines the $SL(2.R)$ group manifold, and the freedom of the Liouville field $q(X)$ (for a given free field $\varphi(X)$) is described only by three parameters of $SL(2.R)$ group.

The solution (2.9) for $\alpha \neq 0$ can be written in the form

$$q(x^+, x^-) = \log \frac{8B'_+(x^+)B'_-(x^-)}{|\Lambda|\left[B_+(x^+)B_-(x^-) - \epsilon(\Lambda)\right]^2}, \qquad \text{where} \quad B_\pm = \alpha F_\pm + \beta_\mp \quad (2.12)$$

The case $\alpha = 0$ is a degenerated one and the corresponding relation between the Liouville and free fields arises for the Backlund transformation [9] (see also [8]).

Note that (2.12) reduces to (1.4) after the substitution $B_\pm \mapsto \pm A_\pm^{\pm 1}$. Thus, (2.9) reproduces the general solution (1.4).

Integrating equations (2.8) with respect to the free field, one can derive the map from the Liouville fields to the free fields

$$\varphi(x^+, x^-) = \log \frac{8G'_+(x^+)G'_-(x^-)}{|\Lambda|[(G_+(x^+) + a_+)(G_-(x^-) + a_-)]^2} + b, \quad (2.13)$$

which is also characterized by three parameters $(a_\pm, b)$ and the functions $G_\pm$ have the form

$$G_+(x^+) = \int_{x_0^+}^{x^+} d\xi^+ e^{q(\xi^+, x_0^+)}, \qquad G_-(x^-) = \int_{x_0^-}^{x^-} d\xi^- e^{q(x_0^+, \xi^-)}. \tag{2.14}$$

The conformal form of the metric tensor (1.1) fixes the gauge freedom only up to the conformal transformations

$$x^\pm \mapsto y^\pm = f^\pm(x^\pm) \tag{2.15}$$

The corresponding transformation of the field q is given by

$$q(x^+, x^-) \mapsto \tilde{q}(x^+, x^-) = q(f^+(x^+), f^-(x^-)) + \log f^{+\,\prime}(x^+) f^{-\,\prime}(x^-). \tag{2.16}$$

it is easy to check that the general solution (1.4) can be obtained by the conformal transformations of the function

$$q(x^+, x^-) = \log \frac{8}{|\Lambda|(x^+ + \epsilon x^-)^2}, \tag{2.17}$$

which is a solution of (1.3).

Therefore, all Liouville fields (with fixed Λ) are related to (2.17) by the conformal transformations (2.16), and hence, by suitable choice of local coordinates one can fix the field $q(X)$. After fixing $q(X)$ the dynamical freedom for the free field $\varphi(X)$ is described only by the parameters $a_\pm, b$ (see (2.13)). As a result, the full gage fixing leads to the finite dimensional system rather than the field theory.

It should be noted that all our construction of this section (as well as the conjecture about the conformal form of the metric tensor (1.1)) is valid only locally.

The maps (2.9) and (2.13) contain singularities when the corresponding denominators are equal to zero. The domain of regularity of q and φ fields can be considered as a patches of the global space-time manifold.

For the global description of the system one has to glue the Liouville (and free) fields given on different patches. In this way we can conclude that the system (2.1) has a finite number of physical degrees of freedom. The number and dynamical character of these parameters are related to the global properties of the space-time manifold.

III Hamiltonian reduction of 2d gravity

In this section we analyze the same problem using the Hamiltonian approach. This approach is non-covariant and we have to separate the space-time coordinates. We assume that the space-time manifold $\mathcal{M}$ can be represented in the form $\mathcal{M} = \mathcal{T} \otimes \mathcal{X}$, where $\mathcal{T}$ is isomorphic to $\mathbf{R}^1$ and $\mathcal{X}$ is a one dimensional manifold. The coordinates x^0 ($x^0 \in \mathcal{T}$) and x^1 ($x^1 \in \mathcal{X}$) we interpret as the time and the space coordinates respectively. We also use the notations $x^0 := t$, $x^1 := x$ and the corresponding derivatives we denote by 'dot' (e.g. $\partial_t \phi := \dot{\phi}$) and 'prime' (e.g. $\partial_x \phi := \phi'$). We assume $g_{00} > 0$ and $g_{11} < 0$.

Due to the reparametrization invariance the Lagrangian of (2.3) is singular and in the Hamiltonian description leads to a constrained dynamical system [5, 6]. For the reduction to the physical (gauge invariant) variables we use the Faddeev-Jackiw's method [10] (see also [11]).

One can check by direct calculation that the above mentioned degeneracy of the Einstein-Hilbert Lagrangian in 2-dimensions can be written in the form

$$\sqrt{-g}R(g_{ab}) = \partial_t \left[\frac{1}{G}\left(\dot{C} - 2B' + B\frac{C'}{C}\right)\right] + \partial_x \left[\frac{1}{G}\left(A' - B\frac{\dot{C}}{C}\right)\right], \tag{3.1}$$

where A, B and C are the components of the metric tensor g_{ab}

$$g_{ab} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \tag{3.2}$$

and

$$G := \sqrt{-g} = \sqrt{B^2 - AC}. \tag{3.3}$$

Then, up to the 'surface terms', the Lagrangian of (2.3) takes the form

$$\mathcal{L} = \frac{1}{2G}\left[C\dot{\phi}^2 - 2B\dot{\phi}\phi' + \frac{B^2}{C}\phi'^{\,2} + 2\dot{\phi}\left(2B' - \dot{C} - B\frac{C'}{C}\right) + 2\phi' B\frac{\dot{C}}{C} - 2\phi'\left(\frac{B^2}{C}\right)'\,\right] - G\left(\Lambda + \frac{2\phi'' C - \phi' C'}{C^2} + \frac{\phi'^{\,2}}{2C}\right), \tag{3.4}$$

where we have excluded the variable A, using relation (3.3). Since the Lagrangian (3.4) does not contain the time derivatives of G and B 'fields', these variables can play a role of Lagrange multipliers.

According to [10] the equivalent system can be described by the action

$$S = \int dx\ dt \left[\Pi_\phi(\dot{\phi} - V_\phi) + \Pi_C(\dot{C} - V_C) + \mathcal{L}(\phi, V_\phi, ...)\right], \tag{3.5}$$

where the Π_ϕ, Π_C, V_ϕ and V_C are auxiliary fields and the function $\mathcal{L}(\phi, V_\phi, ...)$ is constructed from Lagrangian (3.4), substituting the time derivatives $\dot{\phi}$ and $\dot{C}$ by the 'velocities' V_ϕ and V_C respectively.

One can easily eliminate the variables V_ϕ and V_C (taking the corresponding variations) and we arrive at

$$S = \int dx\ dt\ \left[\Pi_\phi\dot{\phi} + \Pi_C\dot{C} + \frac{G}{C}\ \Phi_0 + \frac{B}{C}\Phi_1\right], \tag{3.6}$$

with

$$\begin{aligned} &\Phi_0 = C\Pi_C\Pi_\phi + \frac{1}{2}C^2\Pi_C^2 - \Lambda C - \frac{1}{2}\phi'^{\ 2} - 2\phi'' + \frac{\phi' C'}{C}, \\ &\Phi_1 = \Pi_C C' - \Pi_\phi\phi' + 2C\Pi_C'. \end{aligned} \tag{3.7}$$

The 'fields' G and B are indeed Lagrange multipliers and the corresponding variations give two constraints $\Phi_0 = 0$ and $\Phi_1 = 0$. To analyze this constraint surface it is convenient to introduce the new set of canonically conjugated variables

$$\begin{aligned} &q := \log(-C), \qquad \varphi := \log(-C) - \phi, \qquad (\text{assuming } C < 0) \\ &p := C\Pi_C + \Pi_\phi, \qquad \pi := -\Pi_\phi. \end{aligned} \tag{3.8}$$

Here we use the notations of Sec.II, taking into account that the field $\log(-C)$ in the conformal gauge (1.1) coincides with the field q of the previous section. Note that the conformal gauge (1.1) corresponds to the choice of Lagrange multipliers $(G = -C, B = 0)$, which provides the Liouville and the free equations for q and φ fields respectively.

For the new variables we find

$$\Phi_0 = U_0 - V_0 = 0, \qquad \Phi_1 = U_1 + V_1 = 0, \tag{3.9}$$

where

$$\begin{aligned} &U_0 = \frac{1}{2}(p^2 + q'^{\ 2}) + \Lambda e^q - 2q'', \qquad V_0 = \frac{1}{2}(\pi^2 + \varphi'^{\ 2}) - 2\varphi'', \\ &U_1 = -pq' + 2p', \qquad V_1 = -\pi\varphi' + 2\pi' \end{aligned} \tag{3.10}$$

Note that the functions U_0 and U_1 are T_{00} and T_{01} components of the traceless energy-momentum tensor of the Liouville theory [2, 7, 8], while V_0 and V_1 are the same components for the free theory.

The next step of the reduction procedure [10] is a calculation of the canonical 1-form

$$\theta = \int_{\mathcal{X}} dx\,[p(x)\;dq(x) + \pi(x)\;d\varphi(x)] \tag{3.11}$$

on the constrained surface (3.9). This surface can be represented in the form $\Phi_\pm := U_\pm - V_\pm = 0$, where

$$\begin{aligned} U_\pm &:= \frac{1}{2}\,(U_0 \mp U_1) = \frac{1}{4}(q' \pm p)^2 - (q' \pm p)' + \frac{1}{2}\Lambda e^q, \\ V_\pm &:= \frac{1}{2}\,(V_0 \pm V_1) = \frac{1}{4}(\varphi' \mp \pi)^2 - (\varphi' \mp \pi)'. \end{aligned} \tag{3.12}$$

The canonical 1-form (3.11) defines the canonical commutation relations

$$\{p(x), q(y)\} = \delta(x-y), \quad \{\pi(x), \varphi(y)\} = \delta(x-y), \tag{3.13}$$

and we obtain the Magri brackets [12] for the functions $U_\pm$ and $V_\pm$

$$\begin{aligned} \{U_\pm(x), U_\pm(y)\} &= \mp\left[(U_\pm(x) + U_\pm(y))\,\delta'(x-y) - 2\delta'''(x-y)\right], \\ \{V_\pm(x), V_\pm(y)\} &= \pm\left[(V_\pm(x) + V_\pm(y))\,\delta'(x-y) - 2\delta'''(x-y)\right] \end{aligned} \tag{3.14}$$

and also

$$\{U_+(x), U_-(y)\} = 0, \qquad \{V_+(x), V_-(y)\} = 0. \tag{3.15}$$

As a result, the functions $\Phi_\pm$ are the first class constraints, since they satisfy the commutation relations of the following Lie algebra

$$\{\Phi_\pm(x), \Phi_\pm(y)\} = \mp\,[\Phi_\pm(x) + \Phi_\pm(y)]\,\delta'(x-y). \tag{3.16}$$

The two first class constraints $\Phi_\pm(x)$ usually need two gauge fixing conditions for each point x [6]. Since, at this stage, we have only four fields $(p, q; \pi, \varphi)$, it is expected that the system has no physical degrees of freedom for the field variables. Then, after the full Hamiltonian reduction, the system with only a finite number of degrees of freedom can remain. Below we demonstrate this fact explicitly using again the method [10].

Here it is useful to introduce the functions

$$g_\pm = (q' \mp p - \varphi' \mp \pi)\,e^{-(q \pm \rho)/2}, \qquad \text{with} \quad \rho(x) = \int_{x_0}^{x} d\eta\;\pi(\eta), \tag{3.17}$$

where x_0 is a fixed point from $\mathcal{X}$ ($x_0 \in \mathcal{X}$). Up to the total derivative, we obtain

$$p\dot{q} + \pi\dot{\varphi} \simeq g_-\partial_t e^{(q-\rho)/2} - g_+\partial_t e^{(q+\rho)/2} \simeq e^{(q+\rho)/2}\ \dot{g}_+ - e^{(q-\rho)/2}\ \dot{g}_-. \qquad (3.18)$$

For the restriction of (3.18) on the constrained surface (3.9) we integrate these equations with respect to p and q fields. After integration we get a result, which is similar to (2.9):

$$\begin{aligned} p &= \frac{f''_+}{f'_+} - \frac{f''_-}{f'_-} - 2\frac{\alpha f_- f'_+ - \alpha f_+ f'_- + \beta_+ f'_+ - \beta_- f'_-}{D}, \\ q &= \log\frac{8f'_+ f'_-}{|\Lambda| D^2}, \qquad \text{with} \qquad D := \alpha f_+ f_- + \beta_+ f_+ + \beta_- f_- + \gamma, \end{aligned} \qquad (3.19)$$

where

$$f_\pm(x) = \int_{x_0}^{x} d\xi\ \exp\frac{1}{2}\left[\varphi(\xi) \mp \int_{x_0}^{\xi} d\eta\ \pi(\eta)\right], \qquad (3.20)$$

and the integration constants $(\alpha, \beta_\pm, \gamma)$ satisfy the condition (2.11).

Using (3.17) - (3.20), we find that the restriction of the functions $g_\pm$ on the constrained surface (3.9) is given by

$$g_\pm = - \pm \sqrt{2|\Lambda|}(\alpha f_\pm + \beta_\mp) \qquad (3.21)$$

and (3.18) takes the form

$$p\dot{q} + \pi\dot{\varphi} \simeq \frac{4}{D}\left[f'_+(\alpha\dot{f}_- + \dot{\alpha}f_- + \dot{\beta}_+) - f'_-(\alpha\dot{f}_+ + \dot{\alpha}f_+ + \dot{\beta}_-)\right]. \qquad (3.22)$$

After integration over $\mathcal{X}$, the coefficients of $\dot{\alpha}$ and $\dot{\beta}_\pm$ give the corresponding canonical conjugated variables. The only 'field theory' term is

$$\frac{4\alpha(f'_+\dot{f}_- - f'_-\dot{f}_+)}{D(f_+, f_-)}, \qquad (3.23)$$

but it again gives the 'surface terms', since it is related to the closed 2-form in $f_\pm$ space.

Thus, we get that after the full Hamiltonian reduction there are no degrees of freedom for the field variables. In this way we reproduce the result of Sec.II. To analyze the character of finite dimensional system we need the global description of the model. Such description can be related to the gluing of Liouville fields with singularities (see, for example, [7] and [8]) and we are going to discuss this approach elsewhere.

Acknowledgments

One of the authors (G.J.) is very grateful to the organizers of the seminar 'q-98' for the invitation and kind hospitality.

This work was supported by the grants from: INTAS (96-0482), RFBR (96-01-00344) and the Georgian Academy of Sciences.

References

[1] B. A. Dubrovin, S.P. Novikov and A.T. Fomenko, *Modern Geometry*, (Nauka, Moscow, 1979).

[2] R. Jackiw, *Progress in Quantum Field Theory*, edited by H. Ezawa and S. Kamefuchi (Elsevier Science Publishers B.V., 1986), pp. 83-98;

[3] J. Liouville, J. Math. Pures Appl. **18**, 71 (1853).

[4] N. Seiberg, *Random Surfaces and Quantum Gravity* , edited by O. Alvarez et al. (Plenum Press, New York, 1991) pp. 363-395.

[5] P.A.M. Dirac, *Lectures on Quantum Mechanics.* Belfer Graduate School of Science, (Yeshive University, New York, 1964).

[6] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems.* (Princeton University Press, Princeton, 1992).

[7] G. P. Jorjadze, A. K. Pogrebkov and M. C. Polivanov. Teor. Mat. Fiz., **40**, 221-234, 1979.

[8] G. P. Jorjadze. Teor. Mat. Fiz. **65**, 323- 333, (1985).

[9] G. L. Lamb, Lect. Notes in Math., **515**, 69-79, (1976).

[10] L. Faddeev and R. Jackiw, Phys. Rev. Lett.**60**, 1692 (1988).
R. Jackiw *(Constrained) Quantization Without Tears.* Montepulciano, Italy, (1993).

[11] G. Chechelashvili, G. Jorjadze and N. Kiknadze. Teor. Mat. Fiz., **109**, 90, (1996), or TMF, **109** 1316-1328, (1996).

[12] F. Magri. J.Math. Phys., **19**, 1156-1162, (1978).