Towards the Low Energy Effective Degrees of Freedom in N=2 SUSY Theories: Branes and Integrability

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Abstract

We discuss the brane interpretation of the integrable dynamics behind the exact solution to the N=2 SUSY YM theory. Degrees of freedom in the first integrable system responsible for the spectral Riemann surfaces comes from the hidden Higgs branch of the moduli space. The second integrable system of the Whitham type yields the dynamics on the Coulomb branch and can be considered as the scattering of branes.

The description of the strong coupling regime in the quantum field theory remains a challenging problem and the main hope is connected to discovery of new proper degrees of freedom which would provide the perturbative expansion distinct from the initial one. The first sucsessful derivation of the low energy effective action in N=2 SUSY Yang-Mills theory clearly shows that solution of the theory involves new ingredients which are not familiar in this context before like Riemann surfaces and meromorphic differentials on it [1].

The general structure of the effective actions is defined by the symmetry arguments, in particular they should respect the Ward identies coming from the bare field theory. For example the chiral symmetry fixes the chiral Lagrangian in QCD and the conformal symmetry provides the dilaton

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effective actions in N=0 and N=1 YM theories. Since the effective actions have a symmetry origin one can expect universality properties and generically different UV theories can flow to the same IR ones. It is the symmetry origin of the effective actions that leads to the appearence of the integrable systems on the scene. The point is that the phase spaces for the integrable systems coincide with some moduli space or the cotangent bundle to the moduli space. We can mention KdV hierarchy related to the moduli of the complex structures of the Riemann surfaces, the Toda lattice related to the moduli of the flat connections or Hitchin like systems connected with the moduli of the holomorphic vector bundles. In any case moduli spaces come from some additional symmetry of the problem.

On the other hand to get the effective action one has to integrate over the moduli spaces of the nonperturbative configurations in the theory. Nonperturbative configurations relevant in different dimensions are instantons, monopoles, vortexes or solitons. If one restricts himself to the 4d theories all essential moduli spaces like moduli of flat connections or monopoles can be derived by the reduction proceedure from the universal instanton moduli space. In terms of the integrable systems the problem of calculation of contributions from the nonperturbative fluctuations to the effective action can be reformulated as a calculation of some expectation values in the integrable systems on the moduli space of these fluctuations.

Identification of the variables in the integrable system responsible for some effective action is a complicated problem. At a moment there is no universal way to introduce the proper variables in the theories which are not topological ones but there is some experience in 2d theories [2] which suggests to identify the nonperturbative transition amplitudes among the vacuum states as the dynamical variables. As for the "space-time" variables, coupling constants and sources are the most promising candidates. It is expected that the partition function evaluated in the low-energy effective theory is the so called τ - function in integrable hierarchy which is the generating function for the conserved integrals of motion. The particular solution of the equations of motion in the dynamical system is selected by applying the Ward identities to the partition function of the effective theory.

The arguments above explain the reason for the search of some integrable structures behind the Seiberg-Witten solution to N=2 SUSY Yang-Mills theory. This integrable structures which capture the hidden symmetry structure have been found in [3] where it was shown that A_{N_c} affine Toda chain governs the low energy effective action and BPS spectrum of pure N=2 SYM theory. The generalization to the theories with matter involves Calogero-Moser integrable system for the adjoint matter [4] and XXX spin chain for the fundamental matter [5]. When generalizing to 5d relativistic Toda chain appears to be relevant for the pure gauge theory [6] while anizotropic XXZ chain for SQCD [8]. At the next step completely anisotropic XYZ chain has been suggested as a guide for 6d SQCD [8] while the generalization to the group product case is described by the higher spin magnets [7]. Therefore there are no doubts in the validity of the mapping between effective low-energy efective theories and integrable finite dimensional systems.

The list of correspondences between two seemingly different issues looks as follows. The solution of the classical equation of motion in the integrable system can be expressed in terms of higher henus Riemann surface which can be mapped to the complex Liouville tori of the dynamical system. It is this Riemann surface enters Seiberg-Witten solution, and the meromorphic differential introduced to formulate the solution coincides with the action differential in the dynamical system in the separated variables. The Coulomb moduli space in N=2 theories is identified with the space of the integrals of motion in the dynamical system, for example $Tr\phi^2$ where ϕ is the adjoint scalar field coincides with the Hamiltonian for the periodical Toda system. The parameters of the field theory like masses or Λ_{QCD} determine the parameters and couplings in the integrable system. For instance in SQCD fundamental masses provide the local Casimirs in the periodical spin chains. The full list of interrelations and references can be found in the review [9].

In spite of a lot of supporting facts it is necessary to get more transparent explanation of the origin of integrability in this context. To this aim let us discuss the moduli spaces in the problem at hands. Classically there is only Coulomb branch of the moduli space in pure gauge theory so one can expect dynamical system associated with such phase space. Coulomb branch can be considered as a special Kahler manifold [1] while the Hitchin like dynamical system responsible for the model has a hyperKahler phase space [10]. The resolution of the contradiction comes from the hidden Higgs-like branch which has purely nonperturbative nature [11, 7]. It is the dynamical system on this hidden phase space provides the integrable system of the Hitchin or spin chain type. Therefore there are two moduli spaces in our problem and one expects a pair of dynamical systems. This is what we have indeed; dynamical system on the Higgs branch yields the Hitchin like dynamics with the associated Riemann surfaces while the integrable system on the Coulomb branch gives rise to the Whitham dynamics. The "physical" meaning of the Hitchin system is to incorporate the nonperturbative instanton like contributions to the effective action in the supersymmetric way while the Whitham dynamics is nothing but the RG flows in the model [3].

The next evident question is about the degrees of freedom in both dynamical systems. The claim is that all degrees of the freedom can be identified with the collective coordinates of a particular brane configuration. First let us explain where the Higgs branch comes from. The basic illustrative example for the derivation of the hyperKahler moduli space in terms of branes is the description of ADHM data as a moduli for a system of coupled D1-D5 or D0-D4 branes [14]. If the gauge fields are independent on some dimension one derives Nahm description of the monopole moduli space in terms of D1-D3 branes configuration [15]. The transition from ADHM data to the Nahm ones can be treated as a T duality transformation. At the next step the hyperkahler Hitchin space can be obtained by reducing the dependence (or additional T duality transformation) on one more dimension. This corresponds to the system of D2 branes wrapped around some surface Σ holomorphically embedded in some manifold. The most relevant example concerns T^2 embedded into K3 manifold [16]. The T duality along the torus transforms it to the system of D0 branes on the dual torus, which is the most close picture for the Toda dynamics in terms of D0 branes. The related discussion for the derivation of the Hitchin spaces in terms of instantons on $R^2 \times T^2$ can be found in [17].

Let us now proceed to the explicit brane picture for the N=2 theories. There are different ways to get it, one involves 10d string theory which compactified on the manifold containing the Toda chain spectral curve [18], or the M theory with M5 brane wrapped around the noncompact surface which can be obtained from the spectral curve by deleting the finite number of points [19]. This picture can be considered as the perturbative one and nonperturbative degrees of freedom have to be added. For this purpose it is useful to consider IIA projection of the M theory which involves N_c D4 branes between two NS5 branes located on a distance $\frac{l_s}{g^2}$ along, say x_6 direction. Field theory is defined on D4 branes worldvolume [20] and the extensive review cocerning the derivation of the field theories from branes can be found in [21]. The additional ingredient yielding the hidden Higgs branch comes from the set of N_c D0 branes, one per each D4 brane [11, 7]. It is known that D0 on D4 brane behaves as a abelian point-like instanton but now we have the system of interacting D0 branes. The coupling constant is provided by the Λ_{QCD} parameter which can be most naturally obtained from the mass of the adjoint scalar breaking N=4 to N=2 via dimensional transmutation proceedure.

One way to explain the need for the additional D0,s in IIA theory or KK modes in M theory looks as follows. It is known that any finite-dimensional integrable system with the spectral parameter allows the canonical transformation to the variables – spectral curve with the linear bundle. The spectral curve place is transparent and KK modes provide the linear bundle. As we have already noted they are responsible for the nonperturbative contribution but the summation of the infinite instanton sums into the finite number degrees of freedom remains the challenging problem. It is worth noting that both canonical coordinates in the dynamical system comes from the coordinates of D0 branes in different dimensions. The necessity for the additional nonperturbative degrees of freedom has been also discussed in [12].

To show how the objects familiar in the integrability world translate into the brane language consider two examples. First let us consider the equations of motion in the Toda chain which has the Lax form

$$\frac{dT}{ds} = [T, A] \tag{1}$$

with some $N_c \times N_c$ matrixes T and A. The Lax matrix T can be related to Nahm matrix for the chain of monopoles using the identifications of the spectral curves for cyclic monopole configuration and periodic Toda chain [13]. All these results in the following expression for the Toda Lax operator in terms of the Nahm matrixes T_i

$$T = T_1 + iT_2 - 2iT_3\rho + (T_1 - iT_2)\rho^2 \tag{2}$$

$$\begin{aligned} T_1 &= \frac{i}{2}\sum_{j=1} q_j(E_{+j} + E_{-j}) \\ T_2 &= -\sum_{j=1} q_j(E_{+j} - E_{-j}) \\ T_3 &= \frac{i}{2}\sum_{j} p_j H_j, \end{aligned} \tag{3}$$

where E and H are the standard SU(N) generators, p_i, q_i represent the Toda phase space, and ρ is the coordinate on the CP^1 above. This CP^1 is involved in the twistor construction for monopoles and a point on CP^1 defines the complex structure on the monopole moduli space. With these definitions Toda equation of motion and Nahm equation acquire the simple form

$$\frac{dT}{dt} = [T, A] \tag{4}$$

with fixed A. Having in mind the brane interpretation of the Nahm data [15] we can claim that the equations of motion provide the conditions for the required supersymmetry of the whole configuration.

Given the dynamical system let us discuss the interpretation of the BPS spectrum in the integrable terms [22]. There are many different brane realizations of the BPS spectrum connected by dualities but the "integrable" one can be described in terms of Lax fermions $\Psi(\lambda)$ - eigenfunctions of the Lax operator

$$T\Psi = \lambda\Psi, \tag{5}$$

where λ is the spectral parameter in the dynamical system and simultaneously plays the role of the energy of the spectral fermions. Toda chain spectral curve plays the role of the solution to the equation of motion and simultaneously the dispersion law for the Lax fermions. Therefore it can be shown [22] that the BPS states correspond to the completely filled forbidden or allowed band for the Lax fermion .

As another example of the validity of the brane-integrability correspondence mention the possibility to incorporate the fundamental matter in the gauge theory via branes in two ways. The first one concerns the semiinfinite D4 branes while the second one the set of N_f D6 branes. One can expect two different integrable systems behind and they were found in [23] and [5]. It was shown in [7] that they perfectly correspond to the brane pictures and it appears that the equivalence of two representations agrees with some duality property in the dynamical system. To conclude the discussion of the first dynamical system let us mention that one can inverse the logic and use the possible integrable deformations of the dynamical system to construct their field theory counterparts. Along this line of reasoning we can expect some unusual field theories with the several Λ type scales [7].

Let us proceed now to the second integrable system on the Coulumb branch of the Whitham type. Whitham dynamics provides in a most natural way prepotential $\mathcal{F}$ which yields the low energy effective action in Seiberg-Witten solution. The prepotential as the solution of Whitham equations is related to the action calculated on the solution with Λ_{QCD} playing the role of the time variable. It is illustrative to consider the identity [24]

$$\frac{\partial \mathcal{F}}{\partial log \Lambda} = \beta < Tr\phi^2 > \tag{6}$$

as a relation between action and Hamiltonian, which simultaneously can be treated as the superconformal Ward identity. Remarkably, the prepotential is closely related to the topological 4d theories which can be supported by investigation of the WDVV like equations in 4d [25]. Recently there was some progress concerning the topological properties of N=2 theories which can be formulated in terms of Donaldson theory for the instanton moduli spaces [26, 27]. It appears that this approach is consistent with the Whitham flows in the nonlinear approximation [28] and the second derivative of the prepotential which can be considered as the correlator of the proper operators in N=2 pure gauge theory can be equally calculated within Donaldson [27] and Whitham setups

$$\frac{\partial^2 \mathcal{F}}{\partial T^m \partial T^n} = -\frac{\beta}{2\pi i}\left(\mathcal{H}_{m+1,n+1} + \frac{\beta}{mn}\frac{\partial \mathcal{H}_{m+1}}{\partial a^i}\frac{\partial \mathcal{H}_{n+1}}{\partial a^j}\partial^2_{ij}\log\theta_E(\vec{0}|\mathcal{T})\right) \tag{7}$$

In these formulas the gauge group is $G = SU(N)$, parameter $\beta = 2N$, $m, n = 1, \ldots, N-1$. T_n are the Whitham times and variables a^i are the standard Seiberg-Witten integrals of the meromorphic differential over the i-th cycle on the spectral curve. $\mathcal{H}_{m,n}$ are certain homogeneous combinations of h_k, defined in terms of the h-dependent polynomial $P(\lambda)$ which is used to describe the Seiberg-Witten (Toda-chain) spectral curves:

$$\mathcal{H}_{m+1,n+1} = -\frac{N}{mn}\text{res}_\infty\left(P^{n/N}(\lambda)dP_+^{m/N}(\lambda)\right) = \mathcal{H}_{n+1,m+1} \tag{8}$$

and

$$\mathcal{H}_{n+1} \equiv \mathcal{H}_{n+1,2} = -\frac{N}{n}\text{res}_\infty P^{n/N}(\lambda)d\lambda = h_{n+1} + O(h^2). \tag{9}$$

To recognize the brane realization of the second integrable system of the Whitham type let us adopt slightly different perspective from the F-theory on the elliptically fibered K3 which is equivalent to the orientifold of type IIB theory or, after T duality, to type I theory on T^2. Due to [29] we can treate the N=2 d=4 theory as a world volume theory of 3-branes in the background of the splitted orientifold planes placed at points $\pm\Lambda^2$ in the $u = Tr\phi^2$ complex plane for the SU(2) case. We assume that the possible masses of the fundamental matter tend to infinity so we are in the pure YM case.

Now we have to consider the dynamics of the 3-branes in the directions transverse to the background 7 branes. The arising dynamics is very transparent in the SU(2) case. Let us recall that Whitham dynamics for SU(2) case is governed by the solution of the first Gurevich-Pitaevskii problem [3] which can be easily interpreted as follows. At the initial moment of evolution 3-branes coincide with one of the orientifold planes and with the another planes at the end of the evolution. To analyze the Whitham dynamics in SU(2) case it is convenient to use the following form of the spectral curve

$$y^2 = (x^2 - \Lambda^4)(x - u). \tag{10}$$

The point u represents the position of two 3-branes (which are at $\pm\sqrt{u}$ in the ϕ plane) and another branching points give the fixed positions of the background branes. The branching point u moves according to the Whitham dynamics for the one-gap KdV solution which corresponds to the Seiberg-Witten solution.

$$\eta(x,t) = 2dn^2[\frac{1}{\sqrt{6}}(x - \frac{1+s^2}{3}t, s)] - (1 - s^2) \tag{11}$$

where

$$\frac{1+s^2}{3} - \frac{2s^2(1-s^2)K(s)}{3(E(s) - (1-s^2)K(s))} = \frac{x}{t}, \tag{12}$$

K(s) and E(s) are the elliptic moduli and $s^2 = \frac{u+\Lambda^2}{2\Lambda^2}$. In terms of the automodel variable $\theta = \frac{x}{t}$ the left background brane corresponds to $\theta = -1$ while the right to $\theta = \frac{2}{3}$. Quasiclassical tau-function of this solution provides the prepotential for the SU(2) theory $\mathcal{F}$=log τ_{qcl}. There is no the analogous simple brane picture for the Whitham dynamics for the higher rank groups at a moment.

Therefore we have presented interpretation of the pair of the integrable dynamical systems in the brane terms. It is clear that there are a lot of open questions concerning the integrable structures behind the nonperturbative SUSY YM dynamics. We can mention for instance the spectrum generating algebras in the integrable systems which are expected to be related to the Nakajima,s algebras on the homologies of the instanton moduli spaces, the embedding of the finite dimensional systems as the special solution to the integrable field theories or clarification of the integrable structure behind N=1 theories.

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