# Quantum Moduli Spaces

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**Abstract**

Possible scenario for quantizing the moduli spaces of Riemann curves is considered. The proper quantum observables are quantum geodesics that are invariant with respect to a quantum modular group and satisfy the quantum algebra.

## 1 Introduction

A problem of constructing an appropriate quantum analogue of moduli spaces of various structures related to the higher genus Riemann surfaces drew much attention. In [1], the gauge theory on graphs were associated with the moduli space of flat connections. Amazingly, the moduli spaces of complex structures of Riemann surfaces had not been adequately quantized. The main trouble is the huge discrete symmetry group (the modular group) of the moduli space. So, the moduli spaces are orbifolds of very involved structures. A convenient way to deal with these symmetry groups is to consider a triangulation of the moduli space a la Penner and Kontsevich [2, 3]. This triangulation admits the description in terms of graphs, which we extensively use in the present paper. Besides, in classical case, the Kontsevich matrix model was constructed [3] that generate all observables for 2D quantum gravity with matter.

The question of physical content of such quantum systems remains open. Indeed, already *classical* cohomology theory of moduli spaces of Riemann

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surfaces provides a description of *quantum* topological 2D gravity; therefore, the extra quantization procedure could be related to quantizing the 2D space–time itself.

# 2 Classical moduli spaces

Recall briefly a classical description of Teichmüller spaces of complex structures on Riemann surfaces with holes (punctures).

Teichmüller space $\mathcal{T}^h$ is a space of complex structures on $S$ modulo diffeomorphisms homotopy equivalent to identity. This complex structure inambiguously determines a metric on the Riemann surface, while, in this metric, a neighborhood of each hole corresponds to the exponential mapping from the punctured disc to a half-infinite cylinder.

An oriented 2D surface can be continuously conformally transformed to the constant curvature surface. The Poincaré uniformization theorem claims that any complex surface $S$ of a constant negative curvature (equal $-1$ in what follows) is a quotient of the upper half-plane $\mathbf{H}_+$, endowed with the hyperbolic metric, by a discrete Fuchsian subgroup $\Delta(S)$ of the automorphism group $PSL(2,\mathbf{R})$,

$$S = \mathbf{H}_+/\Delta(S).$$

In the hyperbolic metric, geodesics are either half-infinite circles with endpoints at the real line $\mathbf{R}$ or vertical half-lines; all points of the boundary $\mathbf{R}$ are at infinite distance from each other and from any interior point.

Any hyperbolic homotopy class of closed curves $\gamma$ contains a unique *closed geodesic* of the length $l(\gamma) = |\log \lambda_1/\lambda_2|$, where $\lambda_1$ and $\lambda_2$ are (different) eigenvalues of the element of $PSL(2,\mathbf{R})$ that corresponds to $\gamma$.

## 2.1 Teichmüller space $\mathcal{T}^H(S)$ of surfaces with holes. The graph technique

The central point of the construction is a description of the moduli space $\mathcal{T}^H(S)$ in terms of oriented graphs.

Let a *fat graph* (oriented graph) $\Gamma$ be a graph with the given cyclic ordering of edges entering each (usually, three-valent) vertex. Then, to each (unoriented) edge we put into the correspondence the real number $Z \in \mathbf{R}$;

the set of all such numbers is $\{Z_\alpha | \alpha \in E(\Gamma)\}$ where $E(\Gamma)$ is the set of all (unoriented) edges of the graph $\Gamma$.

**Theorem 1** *For a given three-valent graph $\Gamma$ of the given genus $g$ and number of punctures $n$, there exists a one-to-one correspondence between the set of points of $\mathcal{T}^H(S)$ and the set $\mathbf{R}^{\#\,\mathrm{edges}}$ of edges of this graph supplied with real numbers (lengths).*

We propose the explicit way to construct the Fuchsian group $\Delta(S) \subset PSL(2,\mathbf{R})$ such that $S = \mathbf{H}_+/\Delta(S)$.

To each edge $\alpha$ we associate the matrix $X_{z_\alpha} \in PSL(2,\mathbf{R})$ of the Möbius transform

$$X_{z_\alpha} = \begin{pmatrix} 0 & -\mathrm{e}^{\,Z_\alpha/2} \\ \mathrm{e}^{\,-Z_\alpha/2} & 0 \end{pmatrix}. \tag{2.1}$$

In order to parametrize a *path* over edges of the graph, we introduce the matrices of the "right" and "left" turns

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L \equiv R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \tag{2.2}$$

We introduce now the notion of *geodesic* on the graph.

Let a *closed path* in the graph $\Gamma$ be any oriented path, which starts and terminates at some oriented edge (point) of the corresponding graph. To each such path we put into the correspondence the product of matrices $P_{z_1\cdots z_n} = LX_{z_n}LX_{z_{n-1}}RX_{z_{n-2}}\cdots RX_{z_2}LX_{z_1}$, where the matrices $L$ or $R$ are inserted depending on which turn—left or right—the path is going on the corresponding step.

**Proposition 1** [6] 1. *There is a one-to-one correspondence between the set of all oriented paths $\{P_{z_1\cdots z_n}\}$ and closed (oriented) geodesics $\{\gamma\}$ on the moduli space. Moreover, the length $L(\gamma)$ of a geodesic can be determined as follows:*

$$G(\gamma) \equiv 2\cosh L(\gamma)/2 = \mathrm{tr}\, P_{z_1\cdots z_n}. \tag{2.3}$$

2. *Considering the universal covering $T_\Gamma$ of the graph $\Gamma$, $\Gamma = T_\Gamma/\Delta_\Gamma$, where $\Delta_\Gamma$ is the Fuchsian group of the graph $\Gamma$, we obtain that there is a one-to-one correspondence between the set of all closed geodesics and the primitive conjugacy classes $\{\sigma_i\} \subset \Delta_\Gamma$.*

Summing up, we have the following chain of the one-to-one correspondences:

$$\left\{\begin{array}{c}\text{conjugacy}\\ \text{classes of}\\ \Delta(S)\subset PSL(2,\mathbf{R})\end{array}\right\}\leftrightarrow\left\{\begin{array}{c}\text{closed}\\ \text{geodesics}\\ \text{on } S\end{array}\right\}\leftrightarrow\left\{\begin{array}{c}\text{closed paths}\\ \text{in (any of)}\\ \text{the graph } \Gamma(S)\end{array}\right\}.$$

## 2.2 Weil–Peterson forms

A canonical Poisson structure called the Weil–Peterson structure exists on the set of coordinates $\{Z_\alpha\}$. This structure is degenerate, and the Casimir functions are just the lengths of geodesics surrounding holes. Fixing all these lengths, we obtain a symlpectic leaf. On every such leaf, the Poisson structure can be inverted as to give a symplectic structure. The resulting degenerate two-form is also called the *Weil–Peterson form* [2].

Denote $\alpha(1)$ the oriented edge that is obtained from an oriented edge $\alpha$ by rotating by the angle $2\pi/3$ clockwise around the vertex these edges are starting with and denote $\alpha^{-1}$ the oriented edge that are opposite to an edge $\alpha$. Then, the Weil–Peterson Poisson brackets are

$$\{Z_\alpha,Z_\beta\}=\left\{\begin{array}{rll}1 & \text{if} & \alpha^{\pm1}(1)=\beta \text{ or } \beta^{-1} \qquad \text{and} \quad \alpha^{\mp1}\neq\beta,\beta(1),\\ -1 & \text{if} & \alpha^{\pm1}=\beta(1) \text{ or } \beta^{-1}(1) \qquad \text{and} \quad \alpha^{\mp1}\neq\beta,\beta(1),\\ 2 & \text{if} & \alpha^{\pm1}=\beta^{\pm1}(1) \text{ or } \beta^{\mp1}(1),\\ -2 & \text{if} & \alpha^{\pm1}(1)=\beta^{\pm1} \text{ or } \beta^{\mp1},\\ 0 & & \text{otherwise.}\end{array}\right. \tag{2.4}$$

## 2.3 Classical modular transformations

In [6], the graph transformations that preserve Poisson structure (2.4) were obtained. There exists a natural operation called the *flip* or Whitehead move, which corresponds to transitions between graphs. We must determine such a transformation of the variables $Z_\alpha$ that, first, preserve the Poisson structure and, second, after a series of flips that transform a graph to itself, the resulting transformation should be the identity. This implies the presence of another relation called the *pentagon identity*. It appears that the transformations depicted in Fig. 1 (2.5) satisfy both the demands.

In the classical case [6],

$$\begin{aligned} \{A,B,C,D,Z\} &\to \{A+\phi(Z), B-\phi(-Z), C+\phi(Z), D-\phi(-Z), -Z\} \\ &\equiv \{\tilde{A},\tilde{B},\tilde{C},\widetilde{D},\tilde{Z}\}, \qquad (2.5) \\ \phi(Z) &= \log(\,\mathrm{e}^{\,Z}+1). \end{aligned}$$

**Lemma 1** *Transformation* (2.5) *preserves the products over paths, so the classical geodesic length is a modular-invariant function.*

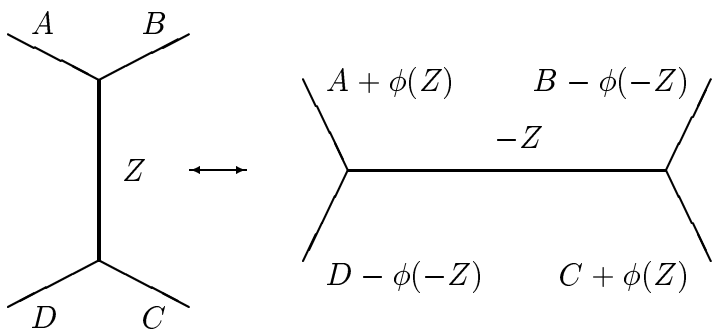



**Fig. 1**

## 2.4 Poisson algebra of geodesics

The functions $\{G_\gamma\}$ (2.3) were studied in [7]. They generate a Poisson algebra (w.r.t. the multiplication and the Weil-Petersson Poisson bracket) over $\mathbf{Z}$. (It means that a product and a Poisson bracket of two such functions is a linear combination of such functions with integer coefficients.)

Consider now the Poisson structure of geodesics. Two nonintersecting geodesics have trivial bracket and, due to the linearity property, we may consider only "simple" intersections of two geodesics $G^1$ and $G^2$ of the form

$$G^1 = \mathrm{tr}^{\,1}\ldots X^1_C R^1 X^1_Z L^1 X^1_A \ldots, \qquad (2.6)$$

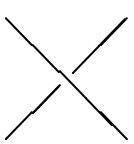
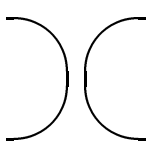
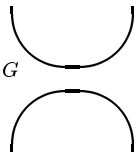
$$G^2 = \mathrm{tr}^{\,2}\ldots X^2_B L^2 X^2_Z R^2 X^2_D \ldots, \qquad (2.7)$$

respectively. (Superscripts 1 and 2 label the matrix spaces.)

A Poisson bracket of geodesics is

$$\{G_1, G_2\} = \frac{1}{2}(\,\mathrm{tr}\,G_{\mathrm{H}} - \,\mathrm{tr}\,G_{\mathrm{I}}), \qquad (2.8)$$

where $G_{\mathrm{I}}$ is the geodesic that is obtained by erasing the edge $Z$ and joining together the edges "$A$" and "$D$" as well as "$B$" and "$C$" in a natural way; the geodesic $G_{\mathrm{H}}$ passes over the edge $Z$ twice, so it has the form $\mathrm{tr}\,\ldots X_C R_Z R_D \ldots X_B L_Z L_A \ldots$. These relations were obtained in [7] in the continuous parameterization (the classical Turaev–Viro algebra), and introducing an additional factor—the total number of geodesics $\#G$—we can uniformly present the classical skein relation as

$(-1)^{\#G}$  $+\,(-1)^{\#G}$  $+\,(-1)^{\#G}$  $= 0.$ (2.9)

# 3 Quantization

Once the graph $\Gamma$ is chosen, the corresponding Teichmüller space $\mathcal{T}^H(S)$ can be easily quantized. Consider the $*$-algebra $\mathcal{T}^\hbar(\Gamma)$ generated by real generators $\{Z_\alpha^\hbar | \alpha \in E(\Gamma)\}$, $(Z_\alpha^\hbar)^* = Z_\alpha^\hbar$ with the defining relations

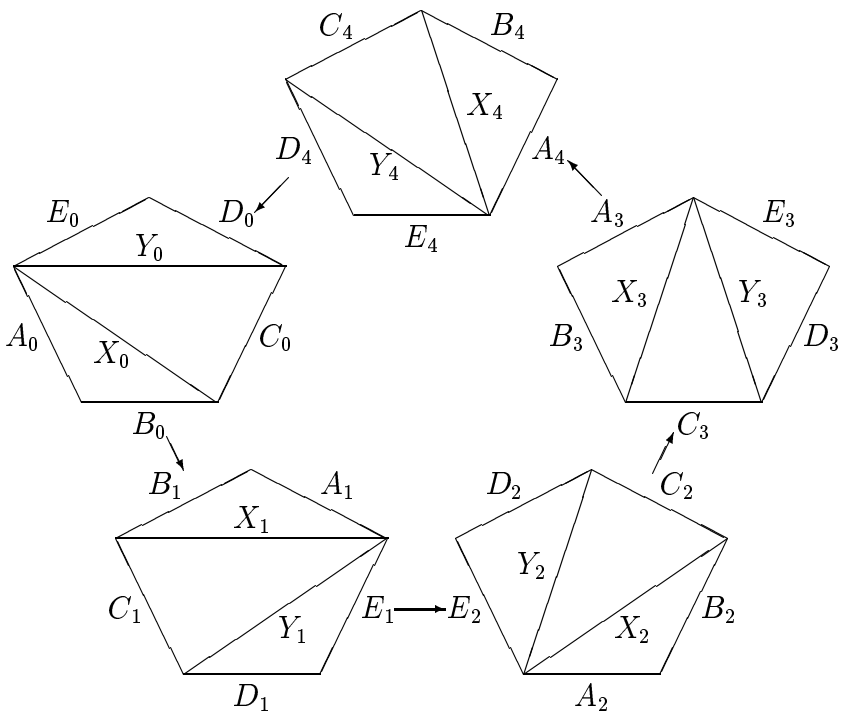
$$[Z_\alpha^\hbar, Z_\beta^\hbar] = 2\pi i\hbar\{Z_\alpha, Z_\beta\} \qquad \text{cf. (2.4).} \tag{3.1}$$

This algebra has an obvious center generated by the perimeters of faces $\left\{P_\gamma | \gamma \in F(\Gamma),\ P_\gamma = \sum_{\alpha\in\gamma} Z_\alpha^\hbar\right\}$. One can easily describe all irreducible $*$-representations of this algebra using the Stone–von Neumann theorem. An irreducible representaion is unambiguously fixed by the values of the operators $P_\gamma$. For example one can represent all operators $Z_\alpha^\hbar$ in $L^2(\mathbf{R}^n)$, where $n = \frac{1}{2}(E(\Gamma) - F(\Gamma))$, by linear combinations with rational coefficients standing by constants and the operators $x_i$ and $i\frac{\partial}{\partial x_i}$, where $\{x_i | i = 1, \ldots, n\}$ is a standard coordinate system on $\mathbf{R}^n$. Now our task is to identify the $*$-algebras constructed using different graphs $\Gamma$ and $\Gamma'$ corresponding to a given surface $S$. In order to make this identification we just construct a $*$-homomorphism $K(\Gamma, \Gamma') : \mathcal{T}^\hbar(\Gamma) \to \mathcal{T}^\hbar(\Gamma')$ of the $*$-algebra generated by $\{Z_{\alpha'}^\hbar | \alpha' \in E(\Gamma')\}$ to the algebra generated by $\{Z_\alpha^\hbar | \alpha \in E(\Gamma)\}$. We require this homomorphism to have the following properties:

1. *Classical limit.* We demand that the algebra homomorphism should tend to the classical homomorphim of the algebras fo function on $\mathcal{T}^H$ when the parameter $\hbar$ tends to zero.

2. *Path independence.* We demand that if we have three graphs $\Gamma$, $\Gamma'$ and $\Gamma''$ then the homomorphisms should satisfy the condition $K(\Gamma'', \Gamma')K(\Gamma', \Gamma) = K(\Gamma'', \Gamma)$.

One can check the latter condition only for one distinguished sequence of flips since others are just compositions of this one. Consider two edges having exactly one common vertex. Then, a sequence of five flips of these edges (such that we never flip the same edge twice consequtively) does not change the graph. This becomed more geometrically transparent if we consider the *dual* graph where the two edges correspond to two edges separating three triangles forming a pentagon. A pentagon can be cut into three triangles in only five possible ways which are related by flips (Fig. 2)




(3.2)

**Fig. 2**

The quantum transforms of the edge variables are as in Fig. 1 with the

function

$$\phi(z) = -\frac{\pi\hbar}{2}\int_\Omega \frac{e^{-ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp, \tag{3.3}$$

and the contour $\Omega$ goes along the real axis bypassing the origin from above.

The function (3.3) and $F(x,\hbar)$,

$$F(x,\hbar) := \exp(-\frac{1}{4}\int_\Omega \frac{e^{-ipx}}{p\sinh(\pi p)\sinh(\pi\hbar p)}dp), \tag{3.4}$$

first appeared in [8] possess the following properties.

**1.** $$\lim_{h\to 0}\phi(x,\hbar) = \log(e^x+1) \text{ (classical limit).} \tag{3.5}$$

**2.** $$2\pi i\hbar\frac{\partial}{\partial x}\log F(x,\hbar) = \phi(x,\hbar). \tag{3.6}$$

**3.** $$F(x+i\pi\hbar,\hbar) = F(x-i\pi\hbar,\hbar)(1+e^x). \tag{3.7}$$

$$F(x+i\pi) = F(x-i\pi)(1+e^{\frac{1}{\hbar}x}). \tag{3.8}$$

**4.** $$\phi(x,\hbar) = \hbar\phi\left(\frac{1}{\hbar}x,\frac{1}{\hbar}\right). \tag{3.9}$$

**Lemma 2** *[4] Quantum transformations* (2.5) *satisfy the pentagon identity.*[1]

## 3.1 Geodesic length operators

The aim of this paragraph is to imbed the algebra of geodesics (2.3) into a suitable completion of the constructed algebra $\mathcal{T}^\hbar(S)$.

The function $G_\gamma$ (2.3) can be expressed for any $\gamma$ in terms of graph coordinates on $\mathcal{T}^H$,

$$G_\gamma \equiv \text{ tr } P_{Z_1\cdots Z_n} = \sum_{j\in J} e^{\frac{1}{2}\sum_{\alpha\in E(\Gamma)} m_j(\gamma,\alpha)z_\alpha}, \tag{3.10}$$

where $m_j(\gamma,\alpha)$ are certain *integer* numbers and $J$ is a finite set of indices. In order to find the *quantum analogues* of these functions, we denote by $\widehat{\mathcal{T}}^\hbar$ a completion of the algebra $\mathcal{T}^\hbar$ containing $e^{xZ_\alpha}$ for any real $x$. Let for any closed path $\gamma$ on $S$, the operator $G^\hbar_\gamma \in \hat{\mathcal{T}}^\hbar$ be

$$G^\hbar_\gamma \equiv {}^\times_\times \text{tr } P_{z_1\ldots z_n} {}^\times_\times = \sum_{\substack{j\in J\\ \kappa\in\{j\}}} e^{\frac{1}{2}\sum_{\alpha\in E(\Gamma)}\left(m_j(\gamma,\alpha)Z^\hbar_\alpha + 2\pi i\hbar c^\kappa_j(\gamma,\alpha)\right)}. \tag{3.11}$$

[1] This result was independently obtained by R. Kashaev [9].

Here some quantum ordering $\overset{\times}{\times}\cdot\overset{\times}{\times}$ is assumed. The main problem is to find the ordering that satisfy all of the conditions below. In (3.11), the numbers $m_j(\gamma,\alpha)$ are the same as in (3.10) and integer coefficients $c_j^\kappa(\gamma,\alpha)$ are to be determined from the procedure of the quantum ordering.

Note that the operators $\{G_\gamma^{\frac{1}{\hbar}}\}$ can be considered as belonging to the algebra $\hat{\mathcal{T}}^\hbar$. In terms of the generators of $\hat{\mathcal{T}}^\hbar$ they are

$$G_\gamma^{\frac{1}{\hbar}} = \sum_{j\in J} e^{\frac{1}{2\hbar}\sum_{\alpha\in E(\Gamma)}\left(m_j(\gamma,\alpha)Z_\alpha^\hbar + 2\pi i c_j^\kappa(\gamma,\alpha)\right)}, \tag{3.12}$$

Now let us formulate the defining properties of quantum geodesics.

**1.** *Modular invariance.* The modular group $\Delta(S)$ (2.5) preserves the set $\{G_\gamma^\hbar\}$, i.e., for any $\delta\in\Delta(S)$ and any closed path $\gamma$, we have $\delta(G_\gamma^\hbar) = G_{\delta\gamma}^\hbar$.

**2.** *Geodesic algebra.* The product of two quantum geodesics is a linear combination of quantum geodesics governed by the skein relation [10].

**3.** *Unorientness.* Quantum traces of direct and inverse geodesic operators coincide.

**4.** *Exponents of geodesics.* Being raised to any power $n$, a quantum geodesics $G_\gamma = 2\cosh L(\gamma)/2$ admits the expansion into the linear combination of quantum geodesics $G_{k\gamma}$, $k=1,\dots,n$; the (binomial) coefficients are the same as in the (classical) relation

$$(2\cosh L(\gamma)/2)^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{k}{n} 2\cosh\Big((n-2k)L(\gamma)/2\Big).$$

**5.** For any $\gamma$ and $\gamma'$, the operators $G_\gamma^\hbar$ and $G_{\gamma'}^{\frac{1}{\hbar}}$ commute.

**6.** If two closed paths $\gamma$ and $\gamma'$ do not intersect, then the operators $G_\gamma^\hbar$ and $G_{\gamma'}^\hbar$ commute.

Let a *simple geodesic* be a geodesic that passes through each edge of a graph no more than once. We denote the Weyl ordering by a usual normal ordering symbol $:\dots:$, i.e.,

$$:\mathrm{e}^{\,a_1}\,\mathrm{e}^{\,a_2}\cdots\mathrm{e}^{\,a_n}: \equiv \mathrm{e}^{\,a_1+\dots+a_n} \quad \text{for any} \quad \{a_i\}.$$

**Lemma 3** *[4] If the quantum transformations* (2.5) *transform a simple geodesic* $G$ *into the simple geodesic* $\widetilde{G}$*, than, for these geodesics, the quantum ordering* $\overset{\times}{\times}\cdot\overset{\times}{\times}$ *is the Weyl ordering and all* $c_j^\kappa(\gamma,\alpha)\equiv 0$ *in* (3.11).

Then we can consider modular transformations that do not preserve the simplicity property and compare the result with the one obtained from the algebraic relations.

The question is which modification of the Weyl ordering produces the proper modular-invariant set of geodesics (3.11)? We partially answer this question in the next section.

## 3.2 Algebra of quantum geodesics

Let $G_1$ and $G_2$ be two simple geodesics with nontrivial intersection. So, for $G_1$ and $G_2$, formula (3.11) implies, by virtue of Lemma 3, the mere Weyl ordering.

After some algebra we have (cf. (2.8))

$$G_1 G_2 = \mathrm{e}^{-i\pi\hbar/2} G_Z + \mathrm{e}^{i\pi\hbar/2} \widetilde{G}_Z, \tag{3.13}$$

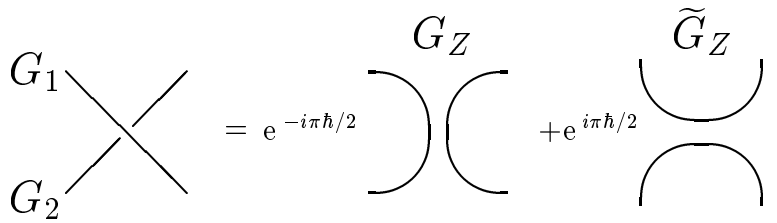
where

$$G_Z = \mathrm{tr}_1 \mathrm{tr}_2 \widehat{\xi}^1_{(ca)} \widehat{\xi}^2(bd)[e^1_{11} \otimes e^2_{22} + e^1_{22} \otimes e^2_{11} - e^1_{12} \otimes e^2_{21} - e^1_{21} \otimes e^2_{12}] \tag{3.14}$$

just like $G_{\mathrm{I}}$ in the classical case (2.8), whereas $\widetilde{G}_Z$ contains the quantum correction term,

$$\begin{aligned} \widetilde{G}_Z &= {}^{\times}_{\times} \mathrm{tr}_1 \mathrm{tr}_2 \widehat{\xi}^1_{(ca)} \widehat{\xi}^2(bd)(e^1_{ij} \otimes e^2_{ji})[X^1_Z \otimes X^2_Z]^{\times}_{\times} \\ &= :\mathrm{tr}_1 \mathrm{tr}_2 \widehat{\xi}^1_{(ca)} \widehat{\xi}^2(bd)(e^1_{ij} \otimes e^2_{ji})[X^1_Z \otimes X^2_Z + 2(1 - \cos\pi\hbar) e^1_{11} \otimes e^2_{22}]: . \end{aligned}$$

Here $e^1_{ij} \otimes e^2_{ji}$ is the standard $r$-matrix that permutes the spaces "1" and "2," so, as a result, the "skein" relation of the form (2.8) appears. Locally, this relation has exactly the form proposed by Turaev [10], i.e., for intersecting simple geodesics, we have the defining relation




(3.15)

(The order of crossing $G_1$ and $G_2$ depends on which geodesic occupies the first place in the product.) Note, however, that assuming the geodesics $G_1$ and $G_2$ to be simple, we may turn the geodesics $\widetilde{G}_Z$ again into the simple geodesics $\widetilde{G}'_Z$ by performing the quantum flip w.r.t. the edge $Z$.

If we now compare two inambiguously defined exrpessions: $\widetilde{G}'_Z$, which must be Weyl ordered, and $\widetilde{G}_Z$ obtained from the geodesic algebra, we find that $\widetilde{G}_Z = \widetilde{G}'_Z$.

**Lemma 4** *The quantum geodesic ordering* $\overset{\times}{\underset{\times}{\ldots}}$ *generated by the geodesic algebra is consistent with the quantum modular transformations* (2.5)*, i.e., the quantum geodesic algebra is modular invariant.*

# 4 Conclusion

In this paper, we briefly described how one may quantize the algebra of geodesics on moduli spaces of complex structures of Riemann surfaces. One may construct all the variety of classical as well as quantum geodesics from a finite number of rather simple objects—the lengths of edges of the graph.

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