Large N_f Calculations in Deep Inelastic Scattering

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Abstract

We describe the evaluation of the anomalous dimensions of twist-2 deep inelastic light cone operators to $O(1/N_f)$ as a check on future perturbative calculations. In particular we present recent results for the singlet gluonic operator dimension in polarised and unpolarised scattering and give three loop predictions for the $O(1/N_f)$ gluonic eigenoperator. The section of the 3-loop $O(1/N_f)$ DGLAP splitting function proportional to the adjoint quadratic Casimir is also calculated for the singlet gluonic operators.

1 Introduction.

Our modern view of hadronic matter is based upon the rich quantum field theory of Quantum Chromodynamics (QCD) (for a review see [1]) in which we picture hadrons as bound states of valence quarks in a sea of vacuum excitation quark/antiquark pairs and gluons. Because QCD has the remarkable property of asymptotic freedom [2] we can use high momentum transfer reactions such as deep inelastic scattering (*lepton* + *nucleon* $\longrightarrow$ *lepton* + *hadrons*) to compare experiment with the predictions of perturbative QCD (pQCD) and so refine our ideas on the structure of strongly interacting particles.

The formal QCD approach to deep inelastic scattering (DIS) allows us to predict structure function behaviour using moment sum rules. Key to this method is the formalism of the light cone expansion (LCE) in which the non-local time ordered product of electromagnetic quark currents appearing in the general DIS cross-section is expanded in a series of local, spin-n operators together with c-number Wilson co-efficients. The LCE is easily seen to be dominated by operators of lowest twist ($\tau =$ *operator dimension* $-$ *operator spin*). For QCD the lowest twist operators available are twist-2. It can be shown by using the anomalous dimensions associated with the renormalisation of insertions of these operators in 2-point Green functions, that we can determine the evolution of the moment sum rules with the large momentum transfer scale.

The unpolarised twist-2 operators are [3],

$$\begin{aligned}
\mathcal{O}^{\mathrm{NS}i}_{\mu_1\ldots\mu_n} &= i^{n-1}\mathcal{S}\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_n}\tfrac{1}{2}\lambda^i\psi \\
\mathcal{O}^{\mathrm{q}}_{\mu_1\ldots\mu_n} &= i^{n-1}\mathcal{S}\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_n}\psi \\
\mathcal{O}^{\mathrm{g}}_{\mu_1\ldots\mu_n} &= \tfrac{1}{2}i^{n-2}\mathcal{S}\ G_{\mu_1\alpha}D_{\mu_2}\ldots D_{\mu_{n-1}}G^{\alpha}{}_{\mu_n}
\end{aligned} \tag{1.1}$$

and the polarised twist-2 operators are [4],

$$\begin{aligned}
\mathcal{R}^{\mathrm{NS}i}_{\sigma\mu_1\ldots\mu_{n-1}} &= i^{n-1}\mathcal{S}\bar{\psi}\gamma_5\gamma_\sigma D_{\mu_1}\ldots D_{\mu_{n-1}}\tfrac{1}{2}\lambda^i\psi \\
\mathcal{R}^{\mathrm{q}}_{\sigma\mu_1\ldots\mu_{n-1}} &= i^{n-1}\mathcal{S}\bar{\psi}\gamma_5\gamma_\sigma D_{\mu_1}\ldots D_{\mu_{n-1}}\psi \\
\mathcal{R}^{\mathrm{g}}_{\sigma\mu_1\ldots\mu_{n-1}} &= \tfrac{1}{2}i^{n-2}\mathcal{S}\ \epsilon_{\sigma\alpha\beta\gamma}G^{\beta\gamma}D_{\mu_1}\ldots D_{\mu_{n-2}}G^{\alpha}{}_{\mu_{n-1}}
\end{aligned} \tag{1.2}$$

Here, ψ is the quark field, $G_{\mu\nu}$ is the gluon field strength tensor, D_μ is the QCD covariant derivative, λ^i are the generators of $SU(N_f)$, $\mathcal{S}$ stands for symmetrisation over Lorentz indices and NS denotes the non-singlet operator under $SU(N_f)$ as opposed to the other singlet operators.

An alternative, less abstract means of providing theoretical predictions for DIS is given by the parton model. Here the structure functions may be expressed in terms of parton distribution functions (PDFs). These give the probability that the struck nucleon constituent (parton) carries a particular fraction of the total nucleon momentum. The evolution of the PDFs are then governed by the DGLAP equation [5]. This contains the DGLAP 'splitting functions' which express the probabilities for the struck parton undergoing certain collinear decays as the energy scale changes. The DGLAP splitting functions can be explicitly obtained from the previously mentioned operator anomalous dimensions through an inverse Mellin transform. Solutions of the full DGLAP equation provide us with scale dependent PDFs which may than be used as inputs for other hard scattering processes where the formal QCD approach is not necessarily applicable as well as giving us a deeper insight into hadron structure in the asymptotic regime.

With facilities such as HERA currently opening up new kinematical regimes for unpolarised and (soon) polarised DIS, there presently exists a need to perform a NNLO twist-2 analysis to ensure a high precision examination of pQCD. This is especially important where the new order's effects are expected to be seen experimentally (as in low x, high Q^2 scattering at HERA). This programme requires the 2-loop Wilson co-efficients and 3-loop anomalous dimensions of all the DIS twist-2 light cone operators with full spin dependence. Although the complete 2-loop operator dimensions have been known for many years now [3, 6, 7, 8], the full 3-loop computations have only become viable in recent years. The current state of play is that the 3-loop unpolarised singlet and non-singlet operator dimensions have been calculated for particular spins of the operator ($n = 2, 4, 6, 8$ and $n = 2, 4, 6, 8, 10$ respectively) [9, 10]. In addition, the 2-loop finite parts required for the full 3-loop results have recently been calculated for all twist-2 unpolarised and polarised operators [11, 12]. As one might expect, these are particularly difficult calculations involving thousands of Feynman diagrams. It is clear that a check on the final full 3-loop results would be useful. One way of approaching this would be to calculate the dimensions using an alternative expansion parameter to the QCD coupling and check that there is agreement where overlap exists. We can do this by use of the $1/N_f$ expansion and a critical point approach.

2 The Method.

By applying the critical large N_f method developed in the series of papers [13] it is possible to obtain expressions for the twist-2 operator dimensions to $O(1/N_f)$. The method entails the analysis of operator insertions in QCD Green functions at a d-dimensional non-trivial renormalisation group fixed point. This fixed point may be found as a stable zero of the d-dimensional four-loop QCD β-function [14] and is located at

$$\begin{aligned} a_c &= \frac{3\epsilon}{4T(R)N_f} + \left[\frac{33}{16}C_2(G)\epsilon - \left(\frac{27}{16}C_2(R) + \frac{45}{16}C_2(G)\right)\epsilon^2 \right. \\ &+ \left(\frac{99}{64}C_2(R) + \frac{237}{128}C_2(G)\right)\epsilon^3 \\ &+ \left. \left(\frac{77}{64}C_2(R) + \frac{53}{128}C_2(G)\right)\epsilon^4 + O(\epsilon^5) \right] \frac{1}{T^2(R)N_f^2} + O\left(\frac{1}{N_f^3}\right) \end{aligned} \tag{2.1}$$

where a_c is the strong coupling constant at criticality, $C_2(R)$ and $C_2(G)$ are the fundamental and adjoint quadratic Casimirs respectively, $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ for T^a the generators of $SU(N_c)$ and $d = 4 - 2\epsilon$.

Several good things come out of using such an approach;

1. Propagators take on a simple, dressed, scaling form due to the scaling properties of Green functions at a renormalisation group fixed point.

   For example, the quark and gluon propagators are respectively [15],

   $$\psi(k) \sim \frac{A\not{k}}{(k^2)^{\mu-\alpha}} \quad , \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu-\beta}}\left[\eta_{\mu\nu} - (1-b)\frac{k_\mu k_\nu}{k^2}\right] \tag{2.2}$$

   where b is the covariant gauge parameter, $d = 2\mu$, A and B are momentum independent amplitudes and the exponents α and β are defined using simple dimensional analysis of the massless QCD action as

   $$\alpha = \mu - 1 + \tfrac{1}{2}\eta \quad , \quad \beta = 1 - \eta - \chi \tag{2.3}$$

   Here, the critical exponent η is the quark anomalous dimension associated with the quark wave function renormalisation at the critical coupling and similarly χ is the anomalous dimension of the quark-gluon vertex.

2. Using the critical renormalisation group it can be shown that the form of a particular $O(1/N_f)$ operator anomalous dimension at a_c may be calculated by evaluating the residues of first order poles (we use an analytic regularisation of the gluon dimension) in appropriate $O(1/N_f)$ two-point Green functions at a_c. These Green functions consist of an insertion of the operator into a QCD two-point function. (We also need to take the field renormalisations into account).

3. Since, in the language of statistical mechanics, we are working at a critical point, we know that we have an essentially massless theory which enables us to use integration tricks such as uniqueness and conformal transformations to evaluate diagrams.

4. In d-dimensions and to $O(1/N_f)$ there exists a universality equivalence between QCD and the non-abelian Thirring model (NATM) at a_c [16]. This may be seen through the reproduction of the QCD triple and quartic gluon vertices by fermion loop integration in three and four point NATM Green functions at a_c using the above propagators. The upshot of this is that we can use the simpler NATM interactions in our calculations and dispense with the tricky three and four-point gluon vertices.

5. The fact that we work with a fixed d-dimensional spacetime with an analytic rather than a dimensional regularisation seems to alleviate some of the problems caused when treating γ_5 in arbitrary dimensions [17]. This obviously becomes important in the calculation of the dimensions of the polarised operators.

3 Comparing singlet operator dimensions at $O(1/N_f)$ with perturbation theory

A slight complication in this calculation arises due to mixing of the singlet operators in both polarised and unpolarised scattering. Since the singlet light cone operators share the same quantum numbers and have equal canonical dimension in strictly four dimensions, they mix under renormalisation. This means that we have to consider a matrix of renormalisation constants for these operators

$$\mathcal{O}^i_{\mathrm{ren}} \;=\; Z^{ij}\mathcal{O}^j_{\mathrm{bare}} \tag{3.1}$$

Here the indices $i,j = q,g$ and we are refering to the operators (1.1) and (1.2). The anomalous dimensions, $\gamma_{ij}(a)$, are defined by

$$\gamma_{ij}(a) \;=\; \left(\begin{array}{cc} \gamma_{qq}(a) & \gamma_{gq}(a) \\ \gamma_{qg}(a) & \gamma_{gg}(a) \end{array} \right) \tag{3.2}$$

where $\gamma_{ij}(a) = \beta(a)(\partial/\partial a)\ln Z_{ij}$ and $\beta(a)$ is the renormalization group function governing the running of the QCD coupling constant a. The entries in $\gamma_{ij}(a)$ depend on the colour group parameters, N_f and n and since it is the $1/N_f$ corrections that we are interested in, we define the explicit form of the entries as

$$
\begin{aligned}
\gamma_{qq}(a) &= a_1 a + (a_{21}\tilde{N}_f + a_{22})a^2 + (a_{31}\tilde{N}_f^2 + a_{32}\tilde{N}_f + a_{33})a^3 + O(a^4) \\
\gamma_{gq}(a) &= b_1 a + (b_{21}\tilde{N}_f + b_{22})a^2 + (b_{31}\tilde{N}_f^2 + b_{32}\tilde{N}_f + b_{33})a^3 + O(a^4) \\
\gamma_{qg}(a) &= c_1\tilde{N}_f a + c_2\tilde{N}_f a^2 + (c_{31}\tilde{N}_f^2 + c_{32}\tilde{N}_f + c_{33})a^3 + O(a^4) \\
\gamma_{gg}(a) &= (d_{11}\tilde{N}_f + d_{12})a + (d_{21}\tilde{N}_f + d_{22})a^2 \\
&\quad +(d_{31}\tilde{N}_f^2 + d_{32}\tilde{N}_f + d_{33})a^3 + O(a^4)
\end{aligned}
\tag{3.3}
$$

where $\tilde{N}_f = T(R)N_f$ and the coefficients a_{ij}, b_{ij}, c_{ij} and d_{ij} depend on n and the colour group Casimirs.

We note that this matrix of anomalous dimensions has eigenvalues

$$
\lambda_{\pm}(a) = \frac{1}{2}(\gamma_{qq} + \gamma_{gg}) \pm \frac{1}{2}\left[(\gamma_{qq} - \gamma_{gg})^2 + 4\gamma_{qg}\gamma_{gq}\right]^{\frac{1}{2}} \tag{3.4}
$$

Expanding in powers of a and retaining the same orders in $1/N_f$ with the definitions (3.3) we find,

$$
\begin{aligned}
\lambda_-(a) &= \left(a_1 - \frac{b_1 c_1}{d_{11}}\right)a + \left(a_{21} - \frac{b_{21}c_1}{d_{11}}\right)N_f a^2 + \\
&\quad \left(a_{31} - \frac{b_{31}c_1}{d_{11}}\right)N_f^2 a^3 + O(N_f^3 a^4) \\
\lambda_+(a) &= \left(d_{11}N_f + d_{12} + \frac{b_1 c_1}{d_{11}}\right)a + \left(d_{21} + \frac{b_{21}c_1}{d_{11}}\right)N_f a^2 \\
&\quad + \left(d_{31} + \frac{b_{31}c_1}{d_{11}}\right)N_f^2 a^3 + O(N_f^3 a^4)
\end{aligned}
\tag{3.5}
$$

Here we can see that $\lambda_+(a)$, $\lambda_-(a)$ are dominated by contributions from the gluonic and fermionic operators respectively.

When in our approach we consider the singlet sector operators in d-dimensions at a_c, we find that they no longer mix. (It is easy to see that the difference is $O(\epsilon)$.) This means that by calculating in d-dimensions, we are accessing the above mixing matrix eigenvalues in perturbation theory. This is borne out by explicit calculations. By evaluating the graphs required for $\gamma_{gg}(a_c)$, $\gamma_{qq}(a_c)$ we actually obtain $\lambda_+(a_c)$ and $\lambda_-(a_c)$ respectively, with the universality equivalence between QCD and NATM at $O(1/N_f)$ accounting for the contributions from the off-diagonal dimensions.

4 Results

The unpolarised non-singlet operator dimension at $O(1/N_f)$ was published in [18]. The polarised non-singlet operator result was published in [19] together with the expressions for $\lambda_-(a_c)$ to $O(1/N_f)$ and $\lambda_+(a_c)$ to $O(1)$ for polarised and unpolarised scattering.

We now give new results [20, 21] for $\lambda_+(a_c)$ to $O(1/N_f)$ for the unpolarised and polarised gluonic operator together with the 3-loop n-dependent eigenvalue predictions for comparison with perturbation theory (3.5). The relevant graphs are given in Fig. 1 with a different operator insertion Feynman rule [7, 22] used for the separate unpolarised and polarised cases.

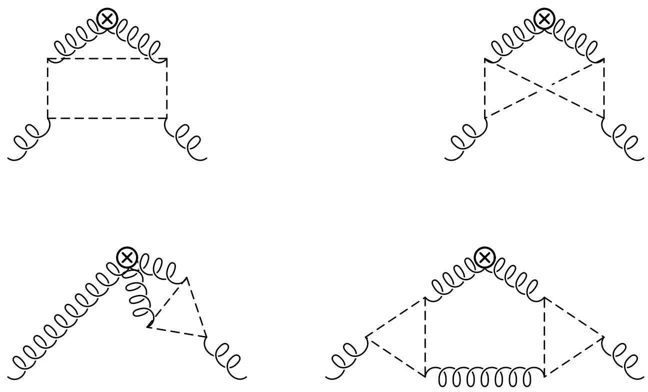

Fig. 1. Leading order diagrams for $\lambda_+(a_c)$.

For the unpolarised gluonic singlet operator we have,

$\lambda_{+,1}(a_c) =$

$$
\begin{aligned}
&- [8\mu^3n^2 + 8\mu^3n + 8\mu^3 + 2\mu^2n^4 + 4\mu^2n^3 - 22\mu^2n^2 - 24\mu^2n \\
&\quad - 28\mu^2 - 6\mu n^4 - 12\mu n^3 + 14\mu n^2 + 20\mu n + 32\mu + 5n^4 \\
&\quad + 10n^3 + n^2 - 4n - 12]\Gamma(n+2-\mu)\Gamma(\mu-1)\mu C_2(R)\eta_1^{\mathrm{O}} \\
&\quad /[(\mu-2)^2(n+2)(n+1)(n-1)\Gamma(2-\mu)\Gamma(\mu+n)nT(R)] \\
&+ \frac{2\mu(\mu-1)S_1(n)C_2(G)\eta_1^{\mathrm{O}}}{(2\mu-1)(\mu-2)T(R)} \\
&- [32\mu^5n^2 + 32\mu^5n + 32\mu^5 - 144\mu^4n^2 - 144\mu^4n - 160\mu^4 - 4\mu^3n^4 \\
&\quad - 8\mu^3n^3 + 240\mu^3n^2 + 244\mu^3n + 316\mu^3 + 16\mu^2n^4 + 32\mu^2n^3 \\
&\quad - 180\mu^2n^2 - 196\mu^2n - 306\mu^2 - 20\mu n^4 - 40\mu n^3 + 59\mu n^2 \\
&\quad + 79\mu n + 146\mu + 8n^4 + 16n^3 - 6n^2 - 14n - 28]\mu C_2(G)\eta_1^{\mathrm{O}} \\
&\quad /[8(2\mu-1)(\mu-1)^3(\mu-2)(n+2)(n+1)(n-1)nT(R)] \\
&+ [32\mu^5n^2 + 32\mu^5n + 32\mu^5 + 8\mu^4n^4 + 16\mu^4n^3 - 120\mu^4n^2 - 128\mu^4n \\
&\quad - 160\mu^4 - 32\mu^3n^4 - 64\mu^3n^3 + 160\mu^3n^2 + 192\mu^3n + 316\mu^3 + 48\mu^2n^4 \\
&\quad + 96\mu^2n^3 - 78\mu^2n^2 - 126\mu^2n - 306\mu^2 - 31\mu n^4 - 62\mu n^3 + 31\mu n \\
&\quad + 146\mu + 7n^4 + 14n^3 + 7n^2 - 28]\Gamma(n+2-\mu)\Gamma(\mu-1)\mu C_2(G)\eta_1^{\mathrm{O}} \\
&\quad /[8(2\mu-1)(\mu-1)^2(\mu-2)(n+2)(n+1)(n-1)\Gamma(2-\mu)\Gamma(\mu+n)nT(R)]
\end{aligned} \tag{4.1}
$$

implying,

$$
\begin{aligned}
&d_{31} + \frac{b_{31}c_1}{d_{11}} = \\
&\quad \frac{64(n^2+n+2)^2(S_1(n))^2C_2(R)}{3(n+2)(n+1)^2(n-1)n^2T(R)} \\
&\quad - \frac{64(10n^6 + 30n^5 + 109n^4 + 168n^3 + 155n^2 + 76n + 12)S_1(n)C_2(R)}{9(n+2)(n+1)^3(n-1)n^3T(R)} \\
&\quad - 4[33n^{10} + 165n^9 - 32n^8 - 1118n^7 - 5807n^6 \\
&\qquad - 12815n^5 - 16762n^4 - 13800n^3 - 7112n^2 \\
&\qquad - 2112n - 288]C_2(R)/[27(n+2)(n+1)^4(n-1)n^4T(R)] \\
&\quad - \frac{8(8n^6 + 24n^5 - 19n^4 - 78n^3 - 253n^2 - 210n - 96)S_1(n)C_2(G)}{27(n+2)(n+1)^2(n-1)n^2T(R)}
\end{aligned}
$$

$$
\begin{aligned}
&- 2[87n^8 + 348n^7 + 848n^6 + 1326n^5 + 2609n^4 + 3414n^3 + 2632n^2 \\
&\quad + 1088n + 192]C_2(G)/[27(n+2)(n+1)^3(n-1)n^3T(R)] \qquad (4.2)
\end{aligned}
$$

For the polarised gluonic singlet operator we have,

$$
\begin{aligned}
\lambda_{+,1}(a_c) = & \\
& - \frac{(n+2)(n-1)\Gamma(n+2-\mu)\Gamma(\mu+1)C_2(R)\eta_1^{\mathrm{O}}}{(\mu-2)^2(n+1)\Gamma(2-\mu)\Gamma(\mu+n)nT(R)} \\
& + \frac{2\mu(\mu-1)S_1(n)C_2(G)\eta_1^{\mathrm{O}}}{(2\mu-1)(\mu-2)T(R)} \\
& - [4\mu^3n^2 + 4\mu^3n - 8\mu^3 - 8\mu^2n^2 - 8\mu^2n + 16\mu^2 + 5\mu n^2 \\
& \quad + 5\mu n - 9\mu - n^2 - n + 2]\Gamma(n+2-\mu)\Gamma(\mu)\mu C_2(G)\eta_1^{\mathrm{O}} \\
& \quad /[8(2\mu-1)(\mu-1)^3(n+1)\Gamma(3-\mu)\Gamma(\mu+n)nT(R)] \\
& - [32\mu^4 - 4\mu^3n^2 - 4\mu^3n - 120\mu^3 + 16\mu^2n^2 + 16\mu^2n \\
& \quad + 160\mu^2 - 20\mu n^2 - 20\mu n - 89\mu + 8n^2 + 8n + 18]\mu C_2(G)\eta_1^{\mathrm{O}} \\
& \quad /[8(2\mu-1)(\mu-1)^3(\mu-2)(n+1)nT(R)] \qquad (4.3)
\end{aligned}
$$

giving,

$$
\begin{aligned}
d_{31} + \frac{b_{31}c_1}{d_{11}} = & \\
& - \frac{64(7n^2+7n+3)(n+2)(n-1)S_1(n)C_2(R)}{9(n+1)^3n^3} \\
& + \frac{64(n+2)(n-1)S_1^2(n)C_2(R)}{3(n+1)^2n^2} \\
& - 4[33n^8 + 132n^7 + 142n^6 - 36n^5 - 263n^4 - 312n^3 \\
& \quad + 280n^2 + 408n + 144]C_2(R)/[27(n+1)^4n^4] \\
& - \frac{8(8n^4+16n^3-19n^2-27n+48)S_1(n)C_2(G)}{27(n+1)^2n^2} \\
& - \frac{2(87n^6+261n^5+249n^4+63n^3-76n^2-64n-96)C_2(G)}{27(n+1)^3n^3} \qquad (4.4)
\end{aligned}
$$

Throughout these calculations we used REDUCE [23] and FORM [24] to handle tedious amounts of algebra. The quantity η_1^{O} is defined by

$$\eta_1^{\mathrm{O}} \;=\; \frac{(2\mu-1)(\mu-2)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)} \tag{4.5}$$

and we have set $\lambda_+(a_c) = \sum_{i=0}^{\infty} \lambda_{+,i}(a_c)/N_f^i$. The finite sum $S_1(n)$ is given by $S_1(n) = \sum_{i=1}^{n} 1/i$.

These expressions agree exactly with all known perturbative results. This can be seen by putting $\mu = 2 - \epsilon$ in (4.2) and (4.4) and expanding in powers of ϵ/N_f. An interesting feature of the mixing matrix (3.2) is that the $O(1/N_f)$ contribution to $\gamma_{gq}(a)$ depends only on the Casimir $C_2(R)$. This is evident from the one and two loop results for all n and the three loop results for $n \leq 8$. Assuming this to be true for all n at three loops we can see from (4.2) and (4.4) that we can deduce the exact form of the coefficients d_{31} in the $C_2(G)$ sector. This means that by using an inverse Mellin transform we can calculate the $O(1/N_f)$ part of the three loop DGLAP function P_{gg} for both unpolarised and polarised cases. For the unpolarised splitting function we obtain,

$$\begin{aligned}
P_{gg}(d_{31}, C_2(G)) &= \frac{1}{4}C_2(G) \\
&\times \left(\frac{64}{27}\left[\frac{1}{1-x}\right]_+ - \frac{64}{27} - \frac{58}{9}\delta(1-x) + \frac{128}{9}(x+1)\mathrm{Li}_2(x) \right. \\
&+ \frac{8}{27}\frac{(x-1)(52x^2+19x+52)}{x}\ln(1-x) - \frac{128}{9}\psi'(1)(x+1) \\
&- \frac{8}{27}(52x^2+43x+76)\ln(x) + \frac{32}{9}(x+1)\ln^2(x) \\
&\left. + \frac{8}{81}\frac{(x-1)(236x^2+47x+236)}{x} \right)
\end{aligned} \tag{4.6}$$

Similarly for the polarised splitting function,

$$\begin{aligned}
P_{gg}(d_{31}, C_2(G)) &= \frac{1}{4}C_2(G) \\
&\times \left(\frac{64}{27}\left[\frac{1}{1-x}\right]_+ - \frac{64}{27} - \frac{58}{9}\delta(1-x) + \frac{128}{9}(x+1)\mathrm{Li}_2(x) \right. \\
&- \frac{128}{9}\psi'(1)(x+1) + \frac{32}{9}(x+1)\ln^2(x) - \frac{8}{27}(67x-56)\ln(x)
\end{aligned}$$

$$-\frac{328}{9}(1-x)\ln(1-x)-\frac{920}{27}(x-1)\Big) \qquad (4.7)$$

These results complete the programme to calculate the $O(1/N_f)$ corrections for the anomalous dimensions of the twist-2 light cone operators. At present it seems that a continuation of this programme to include contributions of $O(1/N_f^2)$ may be viable. We conclude by noting that the results may also be useful in estimating the full three loop corrections to the operator dimensions by using asymptotic Padé approximant techniques [25].

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