# On Large $N$ Conformal Theories, Field Theories in Anti-De Sitter Space and Singletons

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**Abstract**

It was proposed by Maldacena that the large $N$ limit of certain conformal field theories can be described in terms of supergravity on anti-De Sitter spaces ($AdS$). Recently, Gubser, Klebanov and Polyakov and Witten have conjectured that the generating functional for certain correlation functions in conformal field theory is given by the classical supergravity action on $AdS$. It was shown that the spectra of states of the two theories are matched and the two-point correlation function was studied. We consider a model of scalar field with self-interaction and compare the three- and four-point correlation functions computed from a classical action on $AdS$ with the large $N$ limit of conformal theory. An extension of Maldacena's proposal is discussed. We argue that the large $N$ limit of certain conformal field theories in a $p$-brane background can be described in terms of supergravity on the corresponding background. We discuss also the large $N$ limit for the Wilson loop and suggest that singletons which according to Flato and Fronsdal are constituents of composite fields in spacetime should obey the quantum Boltzmann statistics.

# 1 Introduction

The 't Hooft large $N$ limit in QCD where $N$ is the number of colours enables us to understand qualitatively certain striking phenomenological features of strong interactions [1, 2, 3]. To perform an analytical consideration one needs to compute the sum of all planar diagrams. The summation of planar diagrams has been performed only in low dimensional space-times [4].

It was suggested [2] that a master field which dominates the large $N$ limit exists. There was an old problem in quantum field theory how to construct the master field for the large $N$ limit in QCD. This problem has been discussed in many works, for a review see for example [5]. The problem has been reconsidered more recently [6]-[10] by using methods of non-commutative (quantum) probability theory. There are basically two types of correlators in matrix theories. The first type includes trace of operators in different points (one has such correlator, for example in the Wilson loop) and the second one includes the product of traces of the composite local operators. A construction of the master field for the Wilson type correlation functions has been proposed in [8]. It was shown that the master field satisfies to standard equations of relativistic field theory but it is quantized according to the so called quantum Boltzmann relations

$$a_i a_j^* = \delta_{ij}$$

where $a_i$ and $a_j^*$ are annihilation and creation operators. These operators have a realization in the free (Boltzmannian) Fock space. Quantum field theories in Boltzmannian Fock space has been considered in [10].

It was suggested [11] that the Boltzmannian Fock space describing the large $N$ limit of gauge theory should contains states describing black holes which obey the quantum Boltzmann statistics, i.e. black hole can be represented as a Boltzmann gas of branes. This suggestion was based on [8], on the computation of the black hole entropy [12] and on the idea [13] about condensate of D0-branes in the large $N$ limit for the matrix regularisation of membrane. In [14, 15, 16, 17] it was shown how to compute entropy of black hole by using the Boltzmann gas model in Matrix theory [18].

Recently an exciting new development in the study of the large $N$ limit for matrix models has been performed. It was proposed by Maldacena [22] that the large $N$ limit of certain superconformal field theories can be described in terms of supergravity on anti-De Sitter spaces ($AdS$), see [23]-[39]

for further developments. Earlier computations of correlators in the world volume theories are performed in [19, 20, 21]. More recently, Gubser, Klebanov and Polyakov [29] and Witten [31] have conjectured that the generating functional for certain correlation functions in superconformal field theory is given by the classical supergravity action on $AdS$. It was shown that the spectra of states of the two theories are matched and this conjecture was tested for the two-point correlation function.

In this note we discuss the interacting model of the scalar matrix field and compare the three- and four-point correlation functions computed from a classical action on $AdS$ with the large $N$ limit of conformal theory. An extension of Maldacena's proposal is discussed. We argue that the large $N$ limit of certain conformal field theories in a $p$-brane background can be described in terms of supergravity on the corresponding background. We discuss also the large $N$ limit for the Wilson loop and suggest that singletons which according to Flato and Fronsdal [40] are constituents of composite fields in spacetime should obey the quantum Boltzmann statistics.

# 2 Conformal Field Theory

We consider a field $\phi(x)$ in the Euclidean space $R^d$. If one has a transformation law

$$\phi(\lambda x) \to \lambda^{-\Delta}\phi(x) \tag{1}$$

under the scale transformation then the number $\Delta$ is called the (scale) dimension of the field $\phi$. If the value $\Delta$ is canonical then the two-point function is proportional to the free zero mass propagator. To construct a non-trivial conformal invariant field theory we have to assume that at least some of the fields have anomalous dimensions. For a review of conformal field theory see for example [41]. If we have a set of fields $\phi_n(x)$ with dimensions $\Delta_n$ which transform under the infinitesimal conformal transformation

$$\delta x^\mu = x^\mu(\epsilon x) - \frac{1}{2}\epsilon^\mu x^2 \tag{2}$$

as

$$\delta\phi_n(x) = -\Delta_n(\epsilon x)\phi_n(x), \tag{3}$$

then one can derive the two, three and four-point correlation functions in the following known form

$$< \phi(x_1)\phi(x_2) >= \frac{C}{x_{12}^{2\Delta}}, \tag{4}$$

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) >= \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{5}$$

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) >= C_{1234}\prod_{ij} x_{ij}^{\frac{1}{3}\Delta-\Delta_i-\Delta_j} f(\xi,\eta), \tag{6}$$

where $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ and

$$x_{ij} = |x_i - x_j|, \quad \xi = x_{12}x_{34}/x_{13}x_{24}, \quad \eta = x_{12}x_{34}/x_{14}x_{23}.$$

Here $f(\xi,\eta)$ is an arbitrary function. So, conformal invariance defines the two and three-point function up to a constant and the four-point correlation function up to an arbitrary function of two variables. The problem is how to fix the constants $C$'s and the function $f$ by using the known dynamics of the theory.

## 3 The Large $N$ Limit of Matrix Theories

Let us consider the model of an Hermitian scalar matrix field $M(x) = (M_{ij}(x)), \quad i,j = 1,2..N$ in the $d$-dimensional space $R^d$ with the action

$$S = N\int d^dx[\frac{1}{2}Tr(\bigtriangledown M)^2 + \lambda TrM^n]. \tag{7}$$

We are interested in the computation of the large $N$ limit for the following local correlation functions

$$\frac{1}{N} < \mathcal{O}_{i_1}(x_1).....\mathcal{O}_{i_k}(x_k) >, \tag{8}$$

where

$$\mathcal{O}_i(x) = TrM^i(x)$$

as well as the nonlocal (Wilson's type) correlation functions

$$\frac{1}{N} < \mathcal{O}_k(x_1,..x_k) >, \tag{9}$$

where

$$\mathcal{O}_k(x_1,..x_k) = Tr(M(x_1)...M(x_k)). \tag{10}$$

Certainly they are related due to

$$: \mathcal{O}_k(x_1,..x_k) : |_{x_i=x} = \mathcal{O}_k(x). \tag{11}$$

In [8] it was shown that the large $N$ limit of the Wightman functions of the form (9) is governed by the Boltzmann master field. In particular, for the Yang-Mills field $A_\mu(x)$ one has

$$\lim_{N\to\infty} \frac{1}{N} < 0|TrA_{\mu_1}(x_1)....A_{\mu_k}(x_k)|0 >= (\Omega_0|B_{\mu_1}(x_1)...B_{\mu_k}(x_k)|\Omega_0). \tag{12}$$

Here $B_\mu(x)$ satisfies the Yang-Mills equation, but it is quantized according to the Boltzmann commutation relations. For the Wilson loop one has

$$W(C) = \lim_{N\to\infty} \frac{1}{N} < 0|TrPexp\int_C A_\mu dx^\mu|0 >= (\Omega_0|Pexp\int_C B_\mu dx^\mu|\Omega_0). \tag{13}$$

Now we consider a naive but perhaps illuminating extension of the conjecture from [29, 31] to the case of the scalar matrix model (8). One conjectures the following representation for the generating functional of the correlation functions in the large $N$ limit

$$\frac{1}{N} < exp\{\int_{R^d} dx\Phi_0(x)\mathcal{O}(x)\} >= e^{-I(\Phi)} \tag{14}$$

where $\mathcal{O}(x) = TrM^{2d/(d-2)}(x)$. Here $\Phi$ is a field in a $d+1$-dimensional space $B_{d+1}$ such that $R^d$ is its boundary, $\partial B_{d+1} = R^d$. The functional $I(\Phi)$ in (14) is equal to the value of the action for the field $\Phi$ computed on the solution of the corresponding equations of motion with the fixed value $\Phi_0$ on the boundary (i.e. on $R^d$). We assume that the solution $\Phi$ is uniquely defined by the boundary function $\Phi_0$. Therefore $I(\Phi)$ is in fact a functional of $\Phi_0$. In the next section we consider an example of computation of such a functional.

# 4 The Dirichlet Problem for the Non-linear Laplace Equation

General problems of computation of $n$-point correlation functions in the boundary theory are discussed in [31]. Here we carry out an explicit perturbative computation for a model of scalar field with self-interaction. Hopefully this will help to perform more complicated computations in supergravity. The action for the conformal invariant scalar field is

$$I = \int d^d x \sqrt{g}[-\Phi\Delta\Phi - \frac{d-2}{4(d-1)}R\Phi^2 + \lambda\Phi^{2d/(d-2)}] \tag{15}$$

We start with the discussion of the non-linear Laplace equation in the flat space. Let $\Omega$ be an open domain in $R^{d+1}$ and consider the Dirichlet problem

$$\Delta\Phi = \lambda\Phi^{n-1}, \quad x \in \Omega \tag{16}$$

$$\Phi|_{\partial\Omega} = \Phi_0 \tag{17}$$

in the flat metric. Using the Green function $G(x,y)$ satisfying

$$\Delta G(x,y) = -\delta(x-y), \tag{18}$$

$$G(x,y)|_{x\in\partial\Omega} = 0 \tag{19}$$

the solution of the problem (16), (17) can be represented as the solution of the following integral equation [42]

$$\Phi(x) = -\int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n_y}\Phi_0(y)dS_y - \lambda\int_\Omega G(x,y)\Phi^{n-1}(y)dy \tag{20}$$

One can get an expression for $\Phi(x)$ as a functional of $\Phi_0$ by expanding (20) in the perturbation series. Evaluation of the action functional

$$I(\Phi) = \int_\Omega dx\sqrt{g}[\frac{1}{2}(\nabla\Phi)^2 + \frac{\lambda}{n}\Phi^n] \tag{21}$$

can be performed using the representation (20).

Let us consider as a simple example the case of upper half space with the flat metric. We use the following notations. For coordinates in $R^{d+1}$ we use notations $x=(x_0,\mathbf{x})$, $\mathbf{x}=(x_1,...x_d)$, $x^*=(-x_0,\mathbf{x})$, and the upper half space is

$$R^{d+1}_+=\{(x_0,\mathbf{x})\in R^{d+1}|x_0>0\}$$

We also denote

$$|x-y|=\sqrt{(x_0-y_0)^2+|\mathbf{x}-\mathbf{y}|^2},\quad |\mathbf{x}|^2=\sum_{i=1}^{d}x_i^2,\quad |x-\mathbf{y}|=\sqrt{x_0^2+|\mathbf{x}-\mathbf{y}|^2} \tag{22}$$

The Green function for the half space has the form

$$G(x,y)=\frac{1}{b_d}(\frac{1}{|x-y|^{d-1}}-\frac{1}{|x-y^*|^{d-1}}) \tag{23}$$

Using this expression in (20) one gets

$$\Phi(x_0,\mathbf{x})=c_d\int\frac{x_0\Phi_0(\mathbf{y})}{|x-\mathbf{y}|^{d+1}}d\mathbf{y}+$$

$$+\lambda\frac{(c_d)^{n-1}}{b_d}\int dy_0d\mathbf{y}[\frac{1}{|x-y|^{d-1}}-\frac{1}{|x-y^*|^{d-1}}]y_0^{n-1}\prod_{i=1}^{n-1}(\frac{\Phi_0(\mathbf{y}^{(i)})d\mathbf{y}^{(i)}}{|y-\mathbf{y}^{(i)}|^{d+1}})+.... \tag{24}$$

$c_d=\Gamma(\frac{d+1}{2})/\pi^{\frac{d+1}{2}}$. After simple calculations one gets the following representation for the functional $I(\Phi)$

$$I(\Phi)=a_d\int\frac{\Phi_0(\mathbf{x})\Phi_0(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+1}}d\mathbf{x}d\mathbf{y}+\lambda k_d\int dx_0d\mathbf{x}x_0^n\prod_{i=1}^{n}(\frac{\Phi_0(\mathbf{y}^{(i)})d\mathbf{y}^{(i)}}{|x-\mathbf{y}^{(i)}|^{d+1}})+.... \tag{25}$$

The quadratic part includes only integral over the boundary and higher order terms include integration over the bulk (Fig.1). It is interesting to compare these computations with analogous expressions in the functional integral approach to S-matrix [43, 44].

Similar calculations performed for the upper half space with the Lobachevski metric

$$ds^2=\frac{1}{x_0^2}dx^2 \tag{26}$$

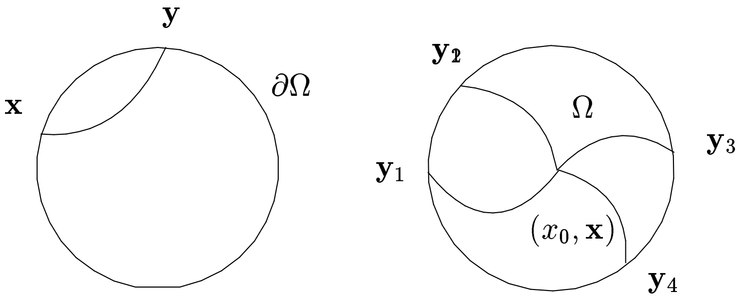


Figure 1: Two- and four-point correlators. Circles represent the boundary $\partial\Omega$ of the domain $\Omega$. The four-point correlator includes an integration over the bulk point $(x_0, \mathbf{x})$

leads to the following effective action

$$I(\Phi) = a_d \int \frac{\Phi_0(\mathbf{x})\Phi_0(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2d}} d\mathbf{x}d\mathbf{y} + \lambda k_d \int dx_0 d\mathbf{x} x_0^{d(n-1)-1} \prod_{i=1}^{n} \left(\frac{\Phi_0(\mathbf{y}^{(i)})d\mathbf{y}^{(i)}}{|x-\mathbf{y}^{(i)}|^d}\right) + .... \tag{27}$$

Expressions (25) and (27) have been obtained by formal manipulations and we ignored divergences. We will present a thorough discussion of these issues in another work [47]. Here we will make only few remarks. To make computations more rigorous it is convenient to use the Fourier transform.

The solution of the Dirichlet problem for the Laplace equation in the upper half space with the flat metric

$$\left(\sum_{i=0}^{d} \frac{\partial^2}{\partial x_i^2}\right)\Phi = 0, \quad \Phi|_{x_0=0} = \Phi_0(\mathbf{x}) \tag{28}$$

can be represented in the form

$$\Phi(x_0, \mathbf{x}) = c \int_{R^d} d\mathbf{p} e^{i\mathbf{p}\mathbf{x} - x_0|\mathbf{p}|} \tilde{\Phi}_0(\mathbf{p}), \tag{29}$$

where

$$\tilde{\Phi}_0(\mathbf{p}) = \int_{R^d} d\mathbf{x} e^{i\mathbf{p}\mathbf{x}} \Phi_0(\mathbf{x}), \tag{30}$$

and we assume that $\Phi_0(\mathbf{x})$ is a test function. Then

$$\frac{\partial \Phi(x_0, \mathbf{x})}{\partial x_0}|_{x_0=0} = -C \int_{R^d} d\mathbf{p} e^{i\mathbf{p}\mathbf{x}} |\mathbf{p}| \tilde{\Phi}_0(\mathbf{p}), \tag{31}$$

and we get for the action

$$I = \frac{1}{2} \int_{R_+^{d+1}} (\nabla \Phi_0(\mathbf{x}.\mathbf{x_0}))^2 dx_0 d\mathbf{x} = \frac{1}{2} \int_{R^d} d\mathbf{x} \Phi_0 \frac{\partial \Phi}{\partial x_0} =$$

$$-\frac{C}{2} \int_{R^d} d\mathbf{x} \Phi_0(\mathbf{x}) \cdot \int_{R^d} d\mathbf{p} e^{i\mathbf{p}\mathbf{x}} |\mathbf{p}| \tilde{\Phi}_0(\mathbf{p}) =$$

$$-\frac{C}{2} \int_{R^d} d\mathbf{p} |\mathbf{p}| |\tilde{\Phi}_0(\mathbf{p})|^2 = \int_{R^d} d\mathbf{x} \Phi(\mathbf{x}) \sqrt{-\Delta} \Phi_0(\mathbf{x}). \tag{32}$$

All these formulae are well defined. $C$ denotes various constants. Now formally one can write the expression (32) as

$$\int d\mathbf{x} d\mathbf{y} \Phi_0(\mathbf{x}) \frac{1}{|\mathbf{x} - \mathbf{y}|^{d+1}} \Phi_0(\mathbf{x}), \tag{33}$$

because

$$\int_{R^d} e^{i\mathbf{p}\mathbf{x}} |\mathbf{p}| d\mathbf{x} = \frac{C}{|\mathbf{x}|^{d+1}}. \tag{34}$$

One interprets (33) as the value of the distribution $|\mathbf{x}-\mathbf{y}|^{-d-1}$ on a test function. The distribution $|\mathbf{x}|^{\lambda}$ is defined by means of the analytical continuation for $\lambda \neq -d, -d-2, -d-4, ...$ [48].

Now let us consider an AdS theory with action

$$I = \frac{1}{2} \int_{\epsilon}^{\infty} dx_0 \int_{R^d} d\mathbf{x} \frac{1}{x_0^{d-1}} \sum_{i=0}^{d} (\frac{\partial \phi}{\partial x_i})^2 \tag{35}$$

Here $\epsilon > 0$ is a cut-off, see [49, 29]. Harmonic analysis on AdS (Lobachevski-Poincare) spaces is considered for example in [50, 51, 52]. The solution of the Dirichlet problem

$$(\sum_{i=0}^{d} \frac{\partial^2}{\partial x_i^2} - \frac{(d-1)}{x_0} \frac{\partial}{\partial x_0}) \Phi = 0, \quad \Phi|_{x_0=0} = \Phi_0(\mathbf{x}) \tag{36}$$

can be represented in the form

$$\Phi(x_0,\mathbf{x}) = c x_0^{d/2} \int_{R^d} d\mathbf{p} e^{i\mathbf{px}} |\mathbf{p}|^{\frac{d}{2}} K_{\frac{d}{2}}(|\mathbf{p}|x_0)\tilde{\Phi}_0(\mathbf{p}), \tag{37}$$

where $K_{\frac{d}{2}}(y)$ is the modified Bessel function. By integrating by parts, one can rewrite (35) as

$$I = -\frac{1}{2}\int_{R^d} d\mathbf{x} (\frac{1}{x_0^{d-1}}\Phi\frac{\partial\Phi}{\partial x_0})|_{x_0=\epsilon} \tag{38}$$

Using the asymptotic expansion of the modified Bessel function one gets a regularised expression for the action.

For $d=4$ one has for $x_0 \to 0$

$$\Phi(x_0,\mathbf{x}) = C\int_{R^4} d\mathbf{p} e^{i\mathbf{px}} [2 - \frac{1}{2}(x_0 p)^2 - \frac{(x_0 p)^4}{8}\ln\frac{x_0 p}{2} + c(x_0 p)^4 + \ ... \ ]\tilde{\Phi}_0(\mathbf{p}), \tag{39}$$

here $p = |\mathbf{p}|$. The action (38) for $\epsilon \to 0$ behaves as

$$I = C\int_{R^4} d\mathbf{p} |\tilde{\Phi}_0(\mathbf{p})|^2 [-\frac{1}{\epsilon^2}p^2 - \frac{p^4}{2}\ln\frac{\epsilon p}{2} + c_1 p^4 + \ ... \ ]. \tag{40}$$

The appearance of divergent terms in the classical action can be related with the fact that the propagator for a field $\mathcal{O}$ of conformal dimension 4 should be a multiple of $|\mathbf{x}-\mathbf{y}|^{-8}$ and one has to define it as a distribution.

In the spirit of the minimal subtractions scheme in the theory of renormalization one can write a "renormalized" action as

$$I_{ren} = C\int_{R^4} d\mathbf{p} |\tilde{\Phi}_0(\mathbf{p})|^2 [-\frac{p^4}{2}\ln\frac{p}{2} + c_1 p^4]. \tag{41}$$

One can write the final result as follows

$$I = \int_{R^5_+} dx\sqrt{g}(\nabla\Phi)^2 \longrightarrow I_{ren} = \int_{R^4} d\mathbf{x} \Phi_0 \Delta^2 [c_1 + c_2 \ln(-\Delta)]\Phi_0 \tag{42}$$

where the arrow includes the renormalization. If one adds also finite parts, then one gets a term $\Phi_0\Delta\Phi_0$. One requires additional physical assumptions to fix the form of the renormalized action.

The renormalized action includes a local term

$$\int_{R^4} d\mathbf{x}(\Delta\Phi_0)^2 \tag{43}$$

There is also a non-local term. This is related with the fact that the distribution $|\mathbf{x}|^{-8}$ [48] is not a homogeneous in $R^4$ (there is *log* in the scaling law).

One can consider the renormalized action (41) as a definition of distribution $|\mathbf{x}-\mathbf{y}|^{-\mathbf{8}}$ and interpret the action [31]

$$\int_{R^{2d}} \frac{\Phi_0(\mathbf{x})\Phi_0(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2d}} d\mathbf{x}d\mathbf{y} \tag{44}$$

as the value of the distribution $|\mathbf{x}-\mathbf{y}|^{-2d}$ on a test function. The Fourier transform of this distribution [48] includes a log-term. For $d=4$ one has

$$\widetilde{|\mathbf{x}|^{-8}} = c_1 p^4 + c_2 p^4 \ln p, \tag{45}$$

which seems is in an agreement with [29].

# 5 Discussion and Conclusion

The supergravity solution in type IIB string theory carrying D3-brane charge has the form

$$ds^2 = f^{-1/2}(-dt^2 + dx_i^2) + f^{1/2} dy_\mu^2$$

where $i = 1,2,3,\ \mu = 4,5,6,7,8,9$ and $f = f(y)$ is a harmonic function. If one has $N$ parallel D3-branes located at $y=0$ then

$$f = 1 + \frac{4\pi g N\alpha'}{|y|^4}$$

In the limit $gN >> 1$ or $y \to 0$ one can neglect the 1 in the harmonic function and the metric describes $AdS_5 \times S^5$ space. The world volume of $N$ parallel D3-branes is described by $\mathcal{N}=4$ supersymmetric $U(N)$ gauge theory in 3+1 dimensions. This is an essential point in the argument leading to the conjecture that type IIB string theory on $(AdS_5 \times S^5)_N$ is dual to super Yang-Mills theory [22].

Now let us consider two bunches of D3-branes located at points $y^{(1)}$ and $y^{(2)}$. In this case

$$f = 1 + 4\pi g\alpha' [\frac{N_1}{|y - y^{(1)}|^4} + \frac{N_2}{|y - y^{(2)}|^4}]$$

There are gravitational forces between two bunches and it seems natural to think that this configuration of branes is described by super Yang-Mills theory in the curved background. In the case $gN_1 >> gN_2 >> 1$ one has super Yang-Mills theory in the 3-brane background. For a previous discussion of M(atrix) theory in curved background along this line see [45, 46]. In the recent paper [39] there is perhaps a related discussion of departures from conformal invariance.

It is known [8] that the large N limit for correlation function of composite operators $\mathcal{O}_k(x_1, ...x_k)$ of the Wilson type (9) is described by the Boltzmann quantum field theory. Due to the relation (11) it seems natural to expect that singletons which are constituents ("partons") of composite fields in conformal theory [40] also should obey the quantum Boltzmann statistics in the large $N$ limit.

The expression (27) obtained for correlation functions in the first order of perturbation theory has an instructive form although we didn't bring it to the conformal invariant form (5), (6). Certainly we have discussed only the model with a simple interaction and one has to look to the more complicated supergravity theory including the Kaluza-Klein modes to check the conjecture [29, 31].

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